Linear Algebra Midterm Sample Questions

Write clearly, with complete sentences, explaining your work. You will be graded on clarity, style, and brevity. If you add false statements to a correct argument, you will lose points.

1. Let V be a vector space over a field F.
   (a) What is the definition of a *linear subspace* of V?
   (b) What is the definition of the *span* of a list $(v_1, \ldots, v_n)$ in a vector space V? Prove that the span of a list in V is the smallest linear subspace of V containing each element of the list.
   (c) What is the definition of the *dimension* of a vector space? Explain why this definition make sense.

2. Let V and W be vector spaces over a field F.
   (a) What is the definition of a *linear transformation* from V to W?
   (b) If $\alpha := (v_1, v_2, \ldots, v_n)$ is a basis for V and $\beta := (w_1, w_2, \ldots w_m)$ is an ordered basis for W, what is the definition of the matrix representation $M^\alpha_\beta(T)$ of a linear transformation from V to W with respect to the bases α and β?
   (c) Let V be the space of polynomials of degree at most 2 over $\mathbf{R}$ and let $\alpha := (1, x, x^2)$, an ordered basis for V. Let $T\colon V \to V$ be the transformation sending p to $p' + 2p$, where p' is the derivative of p. Find $M^\alpha_\alpha(T)$.

3. If V and W are vector spaces, let $\mathcal{L}(V, W)$ denote the set of linear transformations from V to W.
   (a) Explain the definition of the sum $S + T$ of two elements S and T of $\mathcal{L}(V, W)$, and in particular show why, with your definition, $S + T \in \mathcal{L}(V, W)$.

(b) Let $\mathcal{P}$ denote the space of polynomials over the field of real numbers. Explain why the map $D: \mathcal{P} \to \mathcal{P}$ sending f to its derivative is linear. Prove that (Id, D, D^2, D^3) is linearly independent in $\mathcal{L}(\mathcal{P}, \mathcal{P})$.

4. Let V be a finite dimensional vector space and let S and T be linear transformations from V to itself. Prove that if $ST = S + T$, then $ST = TS$. Show that this need not be true if V is not finite dimensional. (Hint: compute $(S - \mathrm{id}_V)(T - \mathrm{id}_V)$.)

5. Let V and W be vector spaces over F and let $V \times W$ be the set of pairs (v, w), where $v \in V$ and $w \in W$. Then $V \times W$ can be made into a vector space using the operations of V and W. We use this structure from now on. If $f: V \to W$ is a function, its graph is the subset of $V \times W$ consisting of those pairs (v, w) such that $w = f(v)$. Show that f is linear if and only if its graph is a linear subspace of $V \times W$.