Traces of operators and matrices

Let $V$ be a finite dimensional $\mathbb{C}$-vector space and let $T$ be an operator on $V$.

Recall:

$$V = \bigoplus \lambda GE_\lambda(T),$$

where $GE_\lambda(T)$ is the generalized $\lambda$-eigenspace of $T$:

$$GE_\lambda(T) = Ker((T - \lambda I)^n) \text{ where } n = \dim V.$$

The characteristic polynomial $f_T$ is the polynomial

$$f_T(t) = \prod_\lambda (t - \lambda)^{d_\lambda} \text{ where } d_\lambda = \dim GE_\lambda.$$

The Cayley-Hamilton theorem:

$$f_T(T) = 0 \text{ as an operator on } V.$$

Goal: compute $f_T$, without actually computing the generalized eigenspaces, or even the eigenvalues.

We know that $f_T(t)$ is monic:

$$f_T(t) = \prod_\lambda (t - \lambda)^{d_\lambda} = t^n + a_1 t^{n-1} + \cdots a_n,$$

for some list of complex numbers $a_i$.

Turns out: $a_1$ is easy to compute. $a_n$ is harder but still possible.

Definition 1 If $T \in \mathcal{L}(V)$ and

$$f_T(t) = t^n + a_1 t^{n-1} + \cdots a_n,$$

is its characteristic polynomial, then

$$\text{trace}(T) := -a_1 \text{ and}$$

$$\det(T) := (-1)^n a_n$$

A partial answer Theorem 8.10:
If $\mathcal{B}$ is a basis for $V$ such that $A := M_\mathcal{B}(T)$ is upper triangular,

$$f_T(t) = \prod_i (t - a_{ii}).$$

However to compute $\mathcal{B}$ we must compute the eigenvalues, which is already difficult.
How does $M_B(T)$ depend on $B$?

Let $B := (v_1, \ldots, v_n)$, $B' := (v'_1, \ldots, v'_n)$ and $B'' := (v''_1, \ldots, v''_n)$ be bases for $V$. If $T_1$ and $T_2$ are operators on $V$, we had the formula

$M_B^B(T_1T_2) = M_B^{B'}(T_1)M_B^{B''}(T_2)$.

Let $S := M_B^B(I)$. (The $j$th column $C_j(S)$ of $S$ is $M_B(v'_j)$.) Apply the formula to see that

$M_B^{B'}(I)M_B^{B''}(I) = M_B^B(I) = I$

Thus $S$ is invertible and $S^{-1} = M_B^B(I)$. Furthermore:

$M_B^{B'}(T) = M_B^{B'}(I)M_B^{B}(T)M_B^{B''}(I) = S^{-1}M_B^{B}(T)S$.

**Example** Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be $T(x_1, x_2) := (x_2, x_1)$. Then in the standard basis $B = (v_1, v_2)$, $T(v_1) = v_2$ and $T(v_2) = v_1$, so

$M_B(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

We also have an eigenbasis $B'$, with $v'_1 = (1,1)$ and $v'_2 = (1,-1)$, so that

$M_B(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note that

$S = M_B^B(I) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$S^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
The trace

Recall that \( \text{trace}(T) = -a_1 \), where

\[
f_r(t) = \prod_{\lambda} (t - \lambda)^{d_\lambda}
\]

Multiply this out to get:

\[
a_1 = -\sum_{\lambda} d_\lambda \lambda
\]

**Definition 2** If \( A \) is an \( n \times n \) matrix,

\[
\text{trace}(A) = \sum_i a_{ii}
\]

**Theorem 3** If \( T \) is an operator on \( V \) and \( B \) is any basis for \( V \),

\[
\text{trace}(T) = \text{trace}(M_B(T)).
\]

Note this is true in the example above, since \( 1 + -1 = 0 + 0 \).

**Proposition 4** If \( A \) and \( B \) are \( n \times n \) matrices,

\[
\text{trace}(AB) = \text{trace}(BA)
\]

**Proof:** Let \( C := AB \) and \( C' := BA \). Then for each \( i \),

\[
c_{ii} = \sum_k a_{ik} b_{ki}
\]

\[
\text{trace}C = \sum_i c_{ii} = \sum_i \sum_k a_{ik} b_{ki}
\]

\[
c'_{ii} = \sum_k b_{ik} a_{ki} = \sum_k a_{ki} b_{ik}
\]

\[
\text{trace}C' = \sum_i c'_{ii} = \sum_i \sum_k a_{ki} b_{ik}
\]

\[
= \sum_k \sum_i a_{ik} b_{ki} = \text{trace}C.
\]

\[\square\]

**Corollary 5** If \( S \) is invertible,

\[
\text{trace}(S^{-1}AS) = \text{trace}(ASS^{-1}) = \text{trace}(A)
\]

**Corollary 6** If \( B \) and \( B' \) are two bases for \( V \),

\[
\text{trace}M_B(T) = \text{trace}M_{B'}(T)
\]

This proves the theorem: Let \( B' \) be a basis of generalized eigenvalues.