Determinants of operators and matrices II

Let V be a finite dimensional **C**-vector space and let T be an operator on V. Recall:

The characteristic polynomial f_T is

$$f_T(t) := \prod_{\lambda}(t-\lambda)^{d_\lambda} = t^n + a_1 t^{n-1} + \cdots a_n, \text{ where } d_\lambda - \dim GE_\lambda.$$

$$\det(T) = \prod_{\lambda} \lambda^{d_\lambda} = (-1)^n a_n$$

Example 1 Cyclic permutations

Let $\mathcal{B} = (v_1, \ldots, v_n)$ and let T be the operator sending v_1 to v_2, v_2 to v_3, and so on, but then v_n to v_1. Then the characterisictic polynomial of T is

$$f_T(t) = t^n - 1 \text{ and}$$

$$\det(T) = (-1)^{n+1}$$

In this example, our linear transformation just *permutes* the basis. Our next step is to discuss more general cases of this.

Permutations

Definition 2 *A permutation of the set* $1, \dots, n$ *is a bijective function* σ *from the set* $\{1, \dots, n\}$ *to itself. Equivalently, it is a list* $(\sigma(1), \dots, \sigma(n))$ *such that each element of* $\{1, \dots, n\}$ *occurs exactly once. The set of all permutations of length* n *is denoted by* S_n*.*

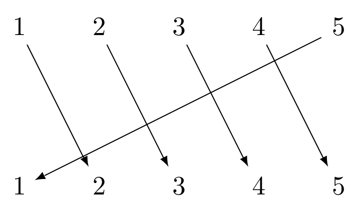
Examples in S_5 :
$(2, 3, 4, 5, 1)$ (cycle of length 5)
$(2, 4, 3, 1, 5)$ (cycle of length 3)
$(2, 4, 5, 1, 3)$ (cycle of length 3 and disjoint cycle of length 2)

Definition 3 *The sign of a permutation* σ *is* $(-1)^m$ *where* m *is the number of pairs* (i, j) *where*

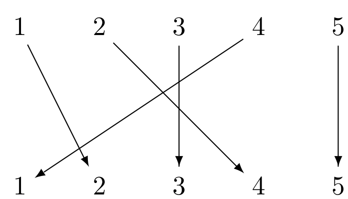
$$1 \leq i < j \leq n \text{ but } \sigma(i) > \sigma(j)$$

Here's an easy way to count: Arrange $(1, 2, \dots, n)$ in one row, and again in a row underneath. $(\sigma(1), \sigma(2), \dots, \sigma(n))$ in a row below. Draw lines connecting i in the first row to $\sigma(i)$ in the second. Then m is the number of crosses.

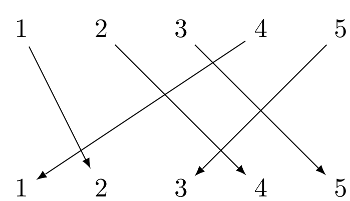
Examples:



so $m = 4$ and sgn $= +1$.



so $m = 4$ and sgn $= +1$.



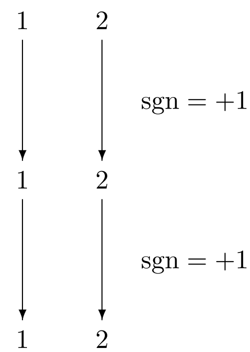
so $m = 5$ and sgn $= -1$.

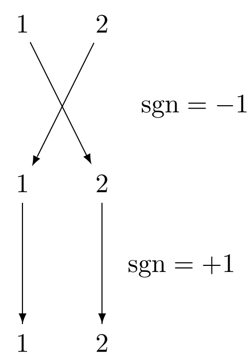
Theorem 4 *If* σ *and* τ *are elemnts of* S_n *and* $\sigma\tau$ *is their composition, then*

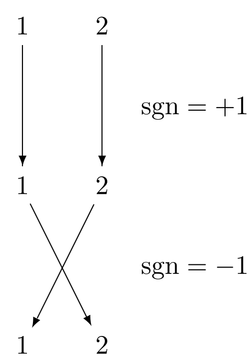
$$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau).$$

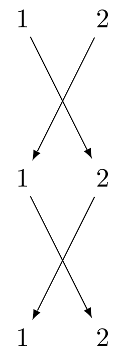
Examples

Let's just look at what happens to one typical pair. There are really four possibilities:









sgn $= -1$

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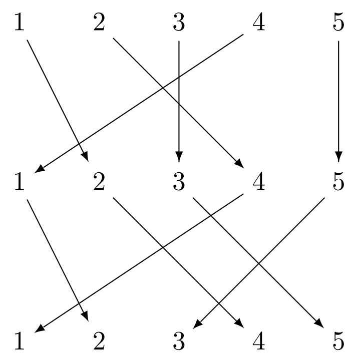
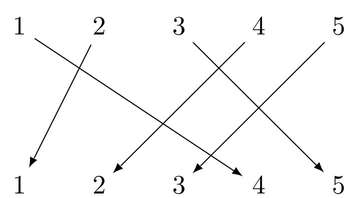
A more complicated example:



so $m = 4$ and sgn $= +1$.

so $m = 5$ and sgn $= -1$.



so $m = 5$ and sgn $= -1$.

Example 5 A cycle of length n has $n-1$ crossings, and so its sign is $(-1)^{n-1}$. Note that this is the same as the determinant of the corresponding linear transfomration.

Definition 6 *Let A be an $n \times n$ matrix. Then*

$$\det A := \sum \{\operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} : \sigma \in S_n\}$$

Example 7 When $n = 2$ there are two permuations, and we get

$$\det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

In general there are $n!$ permuations in S_n, a very big number!

It is often useful to think of a matrix A as a bunch of columns: if A is a matrix, let A_j be its jth column. Then we can think of det as a function of n columns instead of a function of matrices:

$$\det(A) = \det(A_1, A_2, \ldots, A_n)$$

Theorem 8 *Let A and B be $n \times n$ matrices.*

1. *If A is upper triangular,* $\det(A) = \prod_i a_{i,i}$.
2. $\det(A)$ *is a linear function of each column, (when all the other columns are fixed, and similarly for the rows.*
3. *If A' is obtained from A by interchanging two columns, then* $\det(A') = -\det(A)$.
4. *More generally, if A' is obtained from A by a permutation σ of the columns, then* $\det(A') = \operatorname{sgn}(\sigma)\det(A)$.
5. *If two columns of A are equal,* $\det(A) = 0$.
6. $\det(AB) = \det(A)\det(B)$.
7. $\det(A^t) = \det(A)$

Here are some explanations:

1. If A is uppertriangular $a_{ij} = 0$ if $j < i$. Now if $\sigma \in S_n$ is not the identity, $\sigma(i) < i$ for some i, and then $a_{i,\sigma(i)} = 0$. Thus the only term is the sum

   $$\det(A) = \sum_\sigma$$

   is when $\sigma = \text{id}$.

2. This is fairly clear if you think about it. Imagine if $a'_{1j} = ca_{1j}$ for all j, for example.

3. Suppose for example that A' is obtained from A by interchanging the first two columns. Let τ be the permutation interchanging 1 and 2. Then for any j

   $$a'_{i,j} = a_{i,\tau(j)}$$

and for any σ

$$a'_{i,\sigma(i)} = a_{i,\tau\sigma(i)}$$

$$\begin{aligned}
\det A' &:= \sum_\sigma \operatorname{sgn}(\sigma) a'_{1,\sigma(1)} a'_{2,\sigma(2)} \cdots a'_{n,\sigma(n)} \\
&:= \sum_\sigma \operatorname{sgn}(\sigma) a_{1,\tau\sigma(1)} a_{2,\tau\sigma(2)} \cdots a_{n,\tau\sigma(n)} \\
&:= \sum_\sigma -\operatorname{sgn}(\tau\sigma) a_{1,\tau\sigma(1)} a_{2,\tau\sigma(2)} \cdots a_{n,\tau\sigma(n)} \\
&= -\det A
\end{aligned}$$

4. Is proved in exactly the same way.

5. Follows from (3) since then $\det(A) = -\det(A)$.

6. Recall that in fact $B_j = b_{1,j}e_1 + \cdots b_{n,j}e_n$, where e_i is the jth standard basis vector for F^n written as a column. Recall also that if A and B are matrices, then the jth column of AB, which we write as $(AB)j$, is

$$(AB)_j = AB_j = A\sum_i b_{i,j}e_i = \sum_i b_{i,j}Ae_i = \sum_i b_{i,j}A_i$$

So

$$\begin{aligned}
\det(AB) &= \det(AB_1, AB_2, \ldots, AB_n) \\
&= \det(\sum_i b_{i,1}A_i, \sum_i b_{i,2}A_i, \ldots, \sum_i b_{i,n}A_i)
\end{aligned}$$

Using the fact that det is linear with respect to the columns over and over again, we can multiply this out:

$$\det(AB) = \sum_\sigma b_{\sigma(1),1}b_{\sigma(2),2)} \cdots b_{\sigma(n),n)} \det(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)})$$

where here the sum is over all functions σ from the set $\{1, \ldots, n\}$ to itself. But by (5), the determinant is zero if σ is not a permutation, and if it is, we just get the determinant of A times the sign of σ. So (miracle!) we end up with

$$\det(AB) = \sum_\sigma b_{\sigma(1),1}b_{\sigma(2),2)} \cdots b_{\sigma(n),n)}\operatorname{sgn}(\sigma)\det(A) = \det(B)\det(A)$$

7.

$$\begin{aligned}
\det A &:= \sum_\sigma \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\
&= \sum_\sigma \operatorname{sgn}(\sigma^{-1}) a_{1,\sigma^{-1}(1)} a_{2,\sigma^{-1}(2)} \cdots a_{n,\sigma^{-1}(n)} \\
&= \sum_\sigma \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \\
&= \det(A^t)
\end{aligned}$$