Determinants of operators and matrices

Let $V$ be a finite dimensional $\mathbb{C}$-vector space and let $T$ be an operator on $V$.

Recall:

The characteristic polynomial $f_T$ is the polynomial

$$f_T(t) = \prod_{\lambda} (t - \lambda)^{d_{\lambda}} \text{ where } d_{\lambda} = \dim GE_\lambda.$$  

The Cayley-Hamilton theorem:

$$f_T(T) = 0 \text{ as an operator on } V.$$  

Goal: compute $f_T$, without actually computing the generalized eigenspaces, or even the eigenvalues, just from $M_B(T)$ for any arbitrary basis $B$ for $V$.

$$f_T(t) := \prod_{\lambda} (t - \lambda)^{d_{\lambda}} = t^n + a_1 t^{n-1} + \cdots + a_n,$$

for some list of complex numbers $a_i$.

**Definition 1** If $T \in \mathcal{L}(V)$ and

$$f_T(t) = t^n + a_1 t^{n-1} + \cdots + a_n,$$

is its characteristic polynomial, then

$$\text{trace}(T) := -a_1 \text{ and }$$

$$\text{det}(T) := (-1)^n a_n$$

Last time:

If $A := M_B(T)$, then

$$\text{trace}(T) = \text{trace}(A) := \sum a_{i,i}$$
Outline:
1. Define the determinant of a matrix, \( \det(A) \).
2. Check that if \( A \) is uppertriangular,
\[
\det(A) = \prod a_{ii}.
\]
3. Show that if \( A \) and \( B \) are \( n \times n \) matrices,
\[
\det(AB) = \det(A) \det(B).
\]
4. Conclude that \( \det(S^{-1}AS) = \det A \) for every invertible \( S \).
5. Conclude that if \( A = M_B(T) \), then \( \det(A) = \det(T) \).

Step (1) is probably the hardest. How to find the definition? Many approaches. I’ll follow the book, more or less. Let’s look at cyclic spaces.

**Theorem 2** Suppose that \( V \) is \( T \)-cyclic, so that there is a \( v \in V \) with
\[
V = \text{span}(v, Tv, T^2v, \ldots T^{n-1}v).
\]
Then \( (v, Tv, \ldots, T^{n-1}v) \) is a basis of \( V \), and
\[
T^nv = c_0v + c_1Tv + \cdots + c_{n-1}T^{n-1}v
\]
for a unique list \((c_0, \ldots, c_{n-1})\) in \( \mathbb{C} \). Then the characteristic polynomial of \( T \) is
\[
p(t) = t^n - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0.
\]

**Example 3** Last time we looked at \( T(x_1, x_2) := (x_2, x_1) \). Let \( v := (1, 0) \). Then \( (v, Tv) \) is a basis for \( V \), and \( T^2v = v \). Thus \( c_0 = 1 \) and \( c_1 = 0 \), the \( p(t) = t^2 - 1 \).

**Proof:** The equation for \( T^nv \) says that
\[
p(T)(v) = 0.
\]
It follows that
\[
p(T)(T^i v) = T^i p(T)(v) = 0 \text{ for all } i
\]
and since the \( T^i v \)’s span \( V \) \( p(T) = 0 \). Since the list \((v, Tv, \ldots, T^{n-1}v)\) is independent, there is no polynomial of smaller degree that annihilates \( T \). Thus \( p \) is the minimal polynomial of \( T \), and since its degree is the dimension of \( V \) \( p \) is also the characteristic polynomial of \( T \).

**Corollary 4** In the cyclic case above, \( \det(T) = (-1)^{n+1}c_0 \).

**Example 5** Let \( \mathcal{B} = (v_1, \ldots, v_n) \) and let \( T \) be the operator sending \( v_1 \) to \( v_2 \), \( v_2 \) to \( v_3 \), and so on, but then \( v_n \) to \( v_1 \). Then the characteristic polynomial of \( T \) is
\[
f_T(t) = t^n - 1 \text{ and } \det(T) = (-1)^{n+1}
\]
In this example, our linear transformation just *permutes* the basis. Our next step is to discuss more general cases of this.
Permutations

**Definition 6** A permutation of the set $1,\ldots,n$ is a bijective function $\sigma$ from the set $\{1,\ldots,n\}$ to itself. Equivalently, it is a list $(\sigma(1),\ldots,\sigma(n))$ such that each element of $\{1,\ldots,n\}$ occurs exactly once. The set of all permutations of length $n$ is denoted by $S_n$.

Examples in $S_5$:
(2, 3, 4, 5, 1)
(2, 4, 3, 1, 5)
(2, 4, 5, 1, 3)

The first of these is cycle of length 5. Note that the second doesn’t move 3 or 5, and can be viewed as a cyclic permutation of the set $\{1,2,4\}$. The last permutation can be viewed as the product (composition) of a cyclic permutation of $\{1,2,4\}$ and a cyclic permutation of $\{3,5\}$.

**Definition 7** The sign of a permutation $\sigma$ is $(-1)^m$ where $m$ is the number of pairs $(i,j)$ where $1 \leq i < j \leq n$ but $\sigma(i) > \sigma(j)$

Here’s an easy way to count: Arrange $(1,2,\ldots,n)$ in one row, and $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ in a row below. Draw lines connecting $i$ in the first row to $i$ in the second. Then $m$ is the number of crosses.

Examples:

```
1 2 3 4 5
2 3 4 5 1
```

so $m = 4$ and sgn = +1.

```
1 2 3 4 5
2 4 3 1 5
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so $m = 4$ and sgn = +1.

```
1 2 3 4 5
2 4 5 1 3
```

so $m = 5$ and sgn = −1.
Example 8 A cycle of length $n$ has $n - 1$ crossings, and so its sign is $(-1)^{n-1}$. Note that this is the same as the determinant of the corresponding linear transformation.

Theorem 9 If $\sigma, \tau \in S_n$, then

$$\text{sgn}(\sigma \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$$

Omit the proof, at least for now.

Definition 10 Let $A$ be an $n \times n$ matrix. Then

$$\det A := \sum_{\sigma \in S_n} \text{sgn}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$$

Example 11 When $n = 2$ there are two permutations, and we get

$$\det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$ 

Proposition 12 Let $A$ and $B$ be $n \times n$ matrices.

1. If $A$ is upper triangular, $\det(A) = \prod a_{i,i}$.

2. If $B$ is obtained from $A$ by interchanging two columns, then $\det(B) = -\det(A)$.

3. If two columns of $A$ are equal, $\det(A) = 0$.

4. $\det(A)$ is a linear function of each column, (when all the other columns are fixed.)

5. $\det(AB) = \det(A)\det(B)$.