Zeroes and Poles of Zeta functions

The following is a “baby” version of the classic argument of Hadmard and de la Vallée Poussin, later generalized by Deligne in his second proof of the Weil conjectures.

Let \((\alpha_i)\) and \((\beta_i)\) be two finite sequences of complex numbers, with \(\alpha_i \neq \beta_j\) for all \(i\) and \(j\). Let

\[
Z(T) := \prod_{i=1}^\infty \frac{1 - \alpha_i T}{1 - \beta_i T}.
\]

Let

\[
f(T) := \frac{T Z'(T)}{Z(T)} = \sum_i \left( \frac{\beta_i T}{1 - \beta_i T} - \frac{\alpha_i T}{1 - \alpha_i T} \right).
\]

Then

\[
f(T) = \sum_{n \geq 1} a_n T^n, \quad \text{where} \quad a_n = \sum_i \beta_i^n - \alpha_i^n.
\]

Let \(b := \max(|\beta_i|)\) and \(a := \max(|\alpha_i|)\).

**Theorem:** Suppose \(a_n \geq 0\) for all \(n\). Then \(b \geq a\). Furthermore, if \(r\) is the cardinality of the set of \(i\) with \(\alpha_i = b\) and \(s\) is the cardinality of the set of \(i\) with \(|\beta_i| = b\), then \(r \leq s\). If \(s = 1\) and \(|\beta_i| = b\), then in fact \(\beta_i = b\). In this case if also \(r = 1\), and \(|\alpha_j| = b\), then \(\alpha_j = -b\).

**Lemma:** Let \((\lambda_1, \ldots, \lambda_r)\) be a finite sequence of complex numbers of absolute value 1. Then there exists an increasing sequence of natural numbers \((n_k)\) such that \((\lambda_i^{n_k})\) tends to 1 for all \(i\).

Proof: The set of \(\lambda := (\lambda_1, \ldots, \lambda_r)\) is a compact topological space, namely \((S^1)^r\). Hence the sequence \((\lambda^n)\) has a convergent subsequence \((m_j)\). This sequence is Cauchy. So for every number \(i\), there is a number \(K_i\) such that

\[
||\lambda^{m_j} - \lambda^{m_k}|| < 1/i,
\]

whenever \(j\) and \(k\) are at least \(K_i\), (where \(||\cdot||\) means for example the sup norm). Then \(||\lambda^{m_j - m_k} - 1|| < 1/i\) for \(j, k \geq K_i\). In particular, if \(n_1 := m_{K_1+1} - m_{K_1}\), then \(||\lambda^{n_1} - 1|| < 1\). Suppose that \(n_1 < n_2 < \cdots < n_i\) have been chosen so that \(||\lambda^{n_j} - 1|| < 1/j\) for \(j \leq i\). Let

\[
n_{i+1} := m_{K_{i+1}+n_1+1} - m_{K_{i+1}}.
\]

Then \(n_{i+1} > n_i\) and \(||\lambda^{n_{i+1}} - 1|| < 1/(i + 1)\). Thus we have constructed the desired sequence by induction.
To prove the theorem, let \( m := \max(a, b) \). We assume that \( m > 0 \), and dividing by \( m \), we may assume that \( m = 1 \). Let \( r \) be the number of \( i \) such that \( \alpha_i = m \) and \( s \) the number of \( i \) such that \( \beta_i = m \). Then for all \( k \),

\[
a_{nk} = \sum_i \beta_i^{nk} - \alpha_i^{nk} \geq 0.
\]

Taking the limit as \( k \) tends to infinity, we see that each term converges to zero except for those with absolute value 1, which converge to 1. Hence the limit of \( a_{nk} \) is \( s - r \). This implies that \( s - r \geq 0 \), hence that \( b \geq a \).

Now suppose that \( s = 1 \) and \( |\beta_1| = b \). Again we assume without of generality that \( b = 1 \). Assume first that \( r = 0 \). We have \( a_n = \beta_1^n + \epsilon_n \), where \( \epsilon_n \) tends to zero. By the lemma we find a sequence \( (n_k) \) such that \( \beta_1^{nk} \) converge to 1. Then \( \beta_1^{nk+1} \) converges to \( \beta \) and hence \( a_{nk+1} = \beta_1^{nk+1} + \epsilon_{nk+1} \) converges to \( \beta \). Since \( a_{nk+1} \) is always real and nonnegative it follows that \( \beta \) is also. Hence \( \beta = b \). Now suppose that \( r = 1 \). By a similar argument, we find a sequence \( (n_k) \) such that \( \alpha_1^{nk} \) and \( \beta_1^{nk} \) converge to 1. Then

\[
a_{nk+1} = \beta_1^{nk+1} - \alpha_1^{nk+1} + \epsilon_{nk+1}
\]

converges to \( \beta - \alpha \) and

\[
a_{nk+2} = \beta_1^{nk+2} - \alpha_1^{nk+2} + \epsilon_{nk+2}
\]

converges to \( \beta^2 - \alpha^2 \). It follows that \( \beta - \alpha \) and \( \beta^2 - \alpha^2 \) are real and nonnegative. Hence \( \beta + \alpha \) is also real, and hence \( \alpha \) and \( \beta \) are real. Thus \( \beta \) and \( \alpha \) are both plus or minus 1, and \( \alpha = -\beta \). Since \( \beta \geq \alpha \), it is \( \beta \) that is positive.

**Corollary** Let \( X/F_q \) be a smooth, proper, and geometrically connected curve of genus \( g \). Let \( a_n \) be the number of \( F_q^n \)-valued points of \( X/F_q \). Then \( |\alpha_i| < q \) for all \( i \), and there exists an \( r < 1 \) such that

\[
|a_n - (q^n + 1)| \leq 2gq^{nr}
\]

for all \( n \geq 1 \).

Proof: We know that \( a_n = 1 + q^n - \sum \alpha_i^n \) for all \( n \geq 1 \). The theorem says that all \( |\alpha_i| \) are less than \( |q| \), except for the possibility that for one \( i \), \( \alpha_i = -q \). But we know this can’t happen—by making a base change to \( F_{q^2} \), for example. Hence \( |\alpha_i| < q \) for all \( i \). The result follows.

**Corollary** Let \( \pi: X \to Y \) be a separable morphism of degree 2 of smooth projective curves over \( PF_p \). For each \( n \) let \( a_n^+ \) be the cardinality of the set of \( y \in Y(F_p^n) \) with two inverse images in \( X(F_p^n) \) and let \( a_n^- \) be the cardinality
of the set of $y$ with no inverse images, and let $a_n$ be the cardinality of the set of all points in $Y(F_{q^n})$. Then

$$\lim \frac{a_n^+}{a_n} = \lim \frac{a_n^-}{a_n} = 1/2.$$  

Proof: Let $a_n(X)$ be the cardinality of $X(F_{q^n})$. For each $y \in Y(F_{q^n})$, let $c_y$ be the cardinality of $\pi^{-1}(y)$. Then $c_y \in \{0, 1, 2\}$. Furthermore $c_y = 1$ if and only if $y$ is ramified, and the total number $a_n^0$ of such points is uniformly bounded. Hence:

$$a_n(X) = \sum_y c_y = \sum_y (c_y - 1) + a_n(Y) = a_n^+ - a_n^- + a_n(Y)$$

Write $a_n(X) = q^n + \epsilon_n(X)$ and $a_n(Y) = q^n + \epsilon_n(Y)$. Then

$$a_n^+ - a_n^- = \epsilon_n(X) - \epsilon_n(Y)$$

and

$$a_n^+ + a_n^- = a_n(Y) + a_n^0.$$  

Our estimates imply that \(\epsilon_n(X)/a_n(Y)\) and \(\epsilon_n(Y)/a_n(Y)\) tend to zero with $n$, where $\epsilon_n$ means either $\epsilon_n(X)$ or $\epsilon_n(Y)$. Moreover $a_n^0$ is bounded. It follows that

$$\lim \frac{a_n^+ - a_n^-}{a_n} = 0 \text{ and } \lim \frac{a_n^+ + a_n^-}{a_n} = 1.$$  

The theorem follows.