Flatness–a brief overview

Flatness is really an algebraic notion with a subtle geometric interpretation. It is best explained in terms of modules and illustrated by morphisms.

Definition 1 *Let A be a ring and M an A-module. Then M is said to be flat over A if the functor $\otimes M$ for the category of A-modules to itself is exact. It is said to be faithfully flat if it is flat and $M \otimes N = 0$ implies that $N = 0$. An A-algebra B is said to be flat over A if its underlying A-module is flat. If $f: X \to Y$ is a morphism of schemes and G is a sheaf of $\mathcal{O}_X$-modules, then G is said to be flat relative to Y if for every point x in X, the stalk G_x is flat over the local ring $\mathcal{O}_{Y,f(x)}$. The morphism f is flat if $\mathcal{O}_X$ is flat as an $\mathcal{O}_Y$-module.*

We should remark that if B is an A-algebra and N is a B-module, then N is flat as an A-module if and only if for every prime ideal Q of B, N_Q is flat over A_P (where P is the inverse image of Q in A).

Remark 2 *The family of flat morphisms is closed under composition and base change.*

Example 3 A ring A is faithfully flat as a module over itself. Any free A-module is faithfully flat over A. Any localization of A is flat over A. Any direct factor of A (as a ring) is flat over A. A projective A-module is flat over A. If A is an integral domain and I is a proper nonzero ideal, then A/I is not flat over A. A module over a principla ideal domain, or a valuation ring, is flat if and only if it is torsion free. The map $\mathbf{C}[s,t] \to \mathbf{C}[x,y]$ sending s to x and t to xy is not flat.

Remark 4 *If A is a ring and M is an A-module, then it turns out that M is flat if and ony if for every ideal I of A, the natural map*

$$I \otimes_A M \to M$$

is injective. In fact it is enought to check this for for finitely generated ideals (since tensor products commute with direct limits).

Proposition 5 *Let A be a local ring with maximal ideal m and residue field k.*

1. *If M is finitely generated as an A-module, then M is flat iff M is free iff the natural map $m_A \otimes M \to M$ is injective.*
2. *A flat module M is faithfully flat iff $M \otimes k \neq 0$.*
3. *If $A \to B$ is a local homomorphism of noetherian rings and N is a finitely generated B-module, then N is flat over A iff the natural map $m_A \otimes_A N \to N$ is injective.*
4. *Let $A \to B$ be a local homomorphism of local rings. If B is flat, it is faithfully flat and $A \to B$ is injective.*

For example, to prove the injectivity in the last statement, observe that if I is the kernel of $A \to B$, then $I \otimes B = IB = 0$, hence $I = 0$.

A very useful criterion is the criterion of flatness along the fibers.

Proposition 6 *Let $f\colon X \to Y$ be a morphism of flat locally noetherian Z-schemes, where Z is locally noetherian. Let F be a coherent sheaf on X, flat over Z. Then F is flat over Y if and only if its restrictions to the fibers X_z are flat over Y_z.*

Sketch (uses Tor). Let x be a point of X, mapping to $y \in Y$ and $z \in Z$. Let $E^{\cdot}$ be a free resolution of F_x over $\mathcal{O}_{X,x}$. Since F_x is flat relative to $\mathcal{O}_{Z,z}$, the tensor product

$$\overline{E}^{\cdot} := E^{\cdot} \otimes_{\mathcal{O}_{Z,z}} k(z)$$

is a free resolution of $\overline{F}_x := F_x \otimes_{\mathcal{O}_{Z,z}} k(z)$. This is a complex of $\mathcal{O}_{X,z}$-modules, and

$$\overline{E}^{\cdot} \otimes_{\mathcal{O}_{Y_z,y}} k(y) \cong E^{\cdot} \otimes_{\mathcal{O}_{Y,y}} k(y).$$

Since $\overline{F}_x$ is flat over $\mathcal{O}_{Y_z,y}$, the sequence is exact, and hence

$$Tor_1^{\mathcal{O}_{Y,y}}(F_x, k(y)) = 0,$$

proving the flatness.

Corollary 7 *Suppose $X' \to X$ is a closed immersion of locally noetherian flat Z-schemes. If the fibers over Z of the map $X' \to X$ are isomorphisms, so is $X' \to X$.*

Proof: It follows that $X' \to X$ is flat, hence locally an isomorphism. □

Proposition 8 *Let $A \to B$ be a local homomorphism of noetherian local rings and let*

$$u\colon N \to M \to M'' \to 0$$

be an exact sequence of finitely generated B-modules. Suppose M is flat over A and let k be the residue field of A. Then the following are equivalent:

1. *M'' is flat over A and u is injective.*
2. *$u \otimes \mathrm{id}_k$ is injective.*

Proof: Let M' be the image of $N \to M$ and let K be the kernel. Since M is flat, we have an exact sequence:

$$0 \to Tor_1^A(M'', k) \to M' \otimes k \to M \otimes k \to M'' \otimes k \to 0.$$

If (1) holds then $Tor_1^A(M'', k) = 0$, so M'' is flat, and hence so is M'. Since $N \otimes k \to M \otimes k$ is injective, so is the map $N \otimes k \to M' \otimes k$, and hence $N \otimes k \to M' \otimes k$ is bijective. Since M' is flat, $K \otimes k \to N \otimes k$ is injective, hence $K \otimes k = 0$, hence $K = 0$. The converse is clear.

□

Corollary 9 *Let $A \to B$ be a local homomorphism of noetherian local rings. Assume B is flat over A and b is an element of the maximal ideal of B. Then B/bB is flat over A if the image of b in $B \otimes_A k$ is a nonzero divisor.*

Proposition 10 *Let $f: X \to Y$ be a flat morphism of schemes. Then the image of f is closed under generization.*

Proof: Let x be a point of X, let $y := f(x)$ and let η be a point of Y with $y \in \overline{\eta}$. We shall show that there is a generization ξ of x which maps to η. Since every affine neighborhood of y contains η, we may and shall assume that Y is affine. We can also assume that X is affine, so that f corresponds to a homomorphism $\theta: A \to B$. If P corresponds to y and Q to x, we get a local homomorphism of local rings $A_P \to B_Q$. This homomorphism is flat, hence faithfully flat. This implies that $B_Q \otimes_{A_P} k(P')$ is not zero, where P' corresponds to η. But this tensor product corresponds to the fiber of the map $\operatorname{Spec} B_Q \to A_P$ over η. $\square$

Theorem 11 *Let $f: X \to Y$ be a morphism of finite type. Assume Y is locally noetherian. Then X/Y is smooth if and only if it is flat and all its geometric fibers are regular.*

Suppose that the geometric fibers are regular. Then they are smooth. Now suppose x is a point of X and y is its image in Y. Smoothness is local on X, so we may assume that X and Y are affine and that X is a closed subscheme of Z, which is affine n-space over Y. Let I be the ideal of X in Z. Say $Y = \operatorname{Spec} A$, and y corresponds to a prime P of A, and $Z = \operatorname{Spec} B$. Let I be the ideal of B defining X. Since X is flat, the inclusion $I \to B$ induces an incusion $I \otimes A/P \to B/PB$. This says that $I \cap PB = PI$. Now let Q be the ideal of B corresponding to $x \in X \subseteq Z$. The Jacobian criterion for smoothnesss applies to $X_y \to Z_y$ tells us that the map

$$I/(I \cap PB + QI) \to \Omega_{B/A} \otimes k(Q)$$

is injective. But $I \cap PB + QI = PI + QI = QI$, so in fact

$$I/QI \to \Omega_{B/A} \otimes k(Q)$$

is injective. Then X/Y is also smooth in a nbd of x.