Complexes, cones, and triangles

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}(\mathcal{A})$ denote the category of (cochain) complexes in $\mathcal{A}$ and morphims of complexes. Thus an object of $\mathcal{C}$ is a sequence of composable maps:

$$\cdots \to A^{q-1} \xrightarrow{d^{q-1}} A^q \xrightarrow{d^q} A^{q+1\cdots}$$

such that $d^q \circ d^{q-1} = 0$ for all $q \in \mathbf{Z}$. By definition, if $(A^\cdot, d)$ is a complex, the complex $A[m]$ is defined by

$$A[m]^q := A^{q+m}, \quad \text{and} \quad d^q_{A[m]} := (-1)^m d^{q+m}.$$

Note that $(A[m])[n] = A[m+n]$ (this is actually an equality, not an isomorphism!). Let $u\colon A^\cdot \to B^\cdot$ be a morphism in $\mathcal{C}$. Then the *cone* of u is the complex $(C^\cdot(u), d_C)$, where

$$C^q := B^q \oplus A^{q+1} \text{ and } d_C(b,a) := (db - u(a), -da).$$

Some authors use instead the complex with the same terms but with $d_{C'}(b,a) := (db + u(a), -da)$. In fact the map $\sigma\colon (C, d_C) \to (C', d_{C'})$ sending (b,a) to $(b,-a)$ is an isomorphism of complexes.

There is an exact sequence of complexes:

$$0 \longrightarrow B \xrightarrow{u'} C \xrightarrow{u''} A[1] \longrightarrow 0,$$

where $u'(b) := (b,0)$ and $u''(b,a) := a$. Note that there is a commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{u'} & C & \xrightarrow{u''} & A[1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow{u} & B & \xrightarrow{u'} & C' & \xrightarrow{-u'''} & A[1]
\end{array}$$

if u''' in the bottom row is also defined by $u'''(b,a) = a$. Finally, observe that $C(u[1]) = (C'(u))[1]$.

A *triangle* in $\mathcal{C}$ is a sequence of three composable maps:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1].$$

A *morphism of triangles* is a commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
A' & \xrightarrow{u'} & B & \xrightarrow{v'} & C' & \xrightarrow{w'} & A[1]
\end{array}$$

Note that the data of the triangle includes its starting point A. In particular, note that the triangle gives a sequence of maps

$$\cdots \longrightarrow B[-1] \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow A[1] \longrightarrow B[1] \longrightarrow \cdots,$$

but this sequence does not determine the triangle. Observe that if one changes the sign of two of the arrows in a triangle, the new triangle thus obtained is isomorphic to the original triangle, but this is not necessarily the case if one changes one or three of the signs. In particular, if $C(u)$ is the mapping cone of a morphism $u\colon A \to B$, then $C(u[1]) = C'(u)[1]$ and is isomorphic to $C(u)[1]$, but the triangle starting at $A[1]$ deduced from $u[1]$ involves an odd number of sign changes when compared to the triangle obtained by shifting the triangle starting at A deduced from u.

Recall that two maps of complexes $f, g\colon A^{\cdot} \to B^{\cdot}$ are *homotopic* if there exists a family of maps $R^q\colon A^q \to B^{q-1}$ such that $g - f = dR + Rd$.

Definition 1 *Let $\mathcal{K}(\mathcal{A})$ denote the homotopy category of complexes in $\mathcal{A}$. A triangle in $\mathcal{K}$ is distinguished if it is isomorphic to a triangle of the form:*

$$A \xrightarrow{u} B \xrightarrow{u'} C(u) \xrightarrow{u''} A[1]$$

for some morphism $u \in \mathcal{C}(\mathcal{A})$. A triangle is antidistinguished if is obtained from a distinguished triangle by changing the sign of one of the morphisms.

Theorem 2 *The set DT of distinguished triangles in $\mathcal{K}$ satisfied the following conditions.*

1. *(TR1)*

   (a) *Every triangle which is isomorphic to a distinguished triangle is distinguished.*

(b) *Every morphism can be embedded in a triangle.*

(c) *For any* A, $A \xrightarrow{\text{id}} A \longrightarrow 0 \longrightarrow A[1]$ *is distinguished.*

2. *(TR2)*

   (a) *The shift* [1] *of any distinguished triangle is antidistinguished.*

   (b) *A triangle* $A \to B \to C \to A[1]$ *is distinguished if and only if* $B \to C \to A[1] \to B[1]$ *is antidistinguished.*

3. *(TR3) Given two distinguished triangles and a diagram:*

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \end{array}$$

   there is a morphism $h\colon C \to C'$ *(not necessarily unique!) so that* (f, g, h) *is a morphism of triangles.*

4. *(TR4) The octahedral axiom.*

Proof: To prove part (c) of TR1, we have to check that the mapping cone C of the identity map of A is homotopic to 0. In other words, we have to prove that id_C is homotopic to 0. Recall that $C^q = A^q \oplus A^{q+1}$ with $d_C(a, a') = (da - a', -da')$. Then if $R(a, a') := (0, -a)$,

$$(dR + Rd)(a, a') = d(0, -a) + R(da - a', -da') = (a, da) + (a' - da) = (a, a').$$

We have already discussed (a) of TR2. For (b), suppose $u\colon A \to B$ is a morphism in $\mathcal{C}$ and let

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

be the associated distinguished triangle. Let $C(v)$ denote the mapping cone of v. We claim that there is a commutative diagram in $\mathcal{K}$:

$$\begin{array}{ccccccc}
B^{\cdot} & \xrightarrow{v} & C & \xrightarrow{w} & A[1] & \xrightarrow{-u[1]} & B[1] \\
\downarrow \text{id} & & \downarrow \text{id} & & g \downarrow \ \uparrow h & & \downarrow \text{id} \\
B^{\cdot} & \xrightarrow{v} & C & \xrightarrow{v'} & C(v) & \xrightarrow{v''} & B[1]
\end{array}$$

where g and h are inverses of each other. Recall that

$$C(v)^q = C^q \oplus B^{q+1} = B^q \oplus A^{q+1} \oplus B^{q+1},$$

with $d(b, a, b') = (db - u(a) - b', -da', -db')$ and define $g(a) := (0, a, -u(a))$ and $h(b, a, b') := a$. We check that $dh(b, a, b') = -da$ and $hd(b, a, b') = h(db - u(a) - b', -da, -db') = -da$ so h is a morphism. Similarly,

$$dg(a) = d(0, a, -u(a)) = (-u(a) - u(a), -da, du(a)) = (0, -da, du(a))$$

and

$$g(d_{A[1]}a) = g(-d_A a)) = (0, -da, u(da))$$

Note that the diagram commutes. Finally, check that $hg = \text{id}$ and gh is homotopic to the identity, with homotopy given by

$$R(b, a, b') := (0, 0, -b).$$

Conversely, suppose that

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is a triangle such that

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1]$$

is distinguished. Applying what we have just proved twice, we conclude that

$$A[1] \xrightarrow{-u[1]} B[1] \xrightarrow{-v[1]} C \xrightarrow{-w[1]} A[2]$$

is distinguished. Since we have changed the sign three times, it follows that

$$A[1] \xrightarrow{u[1]} B[1] \xrightarrow{v[1]} C \xrightarrow{w[1]} A[2]$$

is antidistinguished, and hence that the original triangle is distinguished.

To prove (3), we may assume that the two triangles come as the cones of morphisms u and u' respectively. By assumption, the square in the diagram is commutative in the homotopy category, so that there is a family of maps $R^q\colon A^q \to B'q-1$ such that $gu = u'f + Rd + dR$. Now define:

$$h\colon C^q \to C'^q : (b,a) \mapsto (g(b) - R(a), f(a))$$

Then it is clear that $f[1]w = w'h$ and that $v'g = hv$. We still need to check that h is a morphism of complexes.

$$\begin{aligned} dh(b,a) &= d(g(b) - R(a), f(a)) \\ &= (dg(b) - dR(a) - u'f(a), -df(a)) \end{aligned}$$

$$\begin{aligned} hd(b,a) &= h(db - u(a), -da) \\ &= (g(db) - gu(a) + Rda, -fda) \end{aligned}$$

Thus it suffices to observe that $-dR(a) - u'f(a) = -gu(a) + Rda$. $\square$