Smooth morphisms

**Definition 1** A morphism \( f : X \to Y \) is smooth if it is locally of finite presentation and formally smooth.

It is clear from the definition that \( I \) is an ideal in an \( R \)-algebra \( A \), then the map \( A^n(A) \to A^n(A/I) \) is surjective. This is in particular true if \( I \) is nilpotent, so that \( A^n \to \text{Spec } R \) is formally smooth, hence smooth.

**Theorem 2** Let \( f : Z \to Y \) be a smooth morphism of schemes, let \( i : X \to Z \) be a closed immersion defined by a sheaf of ideals \( I \), which we assume to be of finite type. Then the following are equivalent:

1. \( X/Y \) is smooth.
2. The map \( I/I^2 \to i^* \Omega_{Z/Y} \) is injective and locally split.

**Proof:** Let \( T \to T' \) be an first order thickening of affine \( Y \)-schemes, with ideal \( I_T, h : T \to X \) be a morphism. Since \( Z/Y \) is smooth, \( h \) can be deformed to a map \( h' : T \to Z \). We need \( h' \) to factor through \( X \), i.e., we need

\[
h'' : I_X \to h_* I_T
\]
to be zero. Since \( I_T^2 = 0 \), this map factors through \( I_X/I_X^2 \). Splitting

\[
\sigma : i^* \Omega_{Z/Y} \to I_X/I_X^2
\]
composed with \( h'' \) gives

\[
\Omega_{Z/Y} \to h_* I_T.
\]
Use this to change the deformation \( h' \) to a new one which works.

For the converse, look at the first infinitesimal nbd. \( X_1 \) of \( X \) in \( Z \). Smoothness gives a deformation \( X_1 \to X \), and we use this to get a section as before.

**Corollary 3** If \( X/Y \) is smooth, \( \Omega_{X/Y} \) is locally free.

**Corollary 4** Let \( Z/Y \) be a smooth morphism and let \( i : X \to Z \) be a closed immersion with ideal \( I \), and let \( x \) be a point of \( X \). Then the following are equivalent:

1. There is an open neighborhood \( U \) of \( x \) which is smooth over \( Y \).
2. The map \( I(x) \to \Omega_{Z/Y}(x) \) induced by \( d \) is injective.

**Proof:** Suppose (2) holds. Choose a basis for the \( k \)-vector space \( I/mI \) and lift it to a sequence \((a_1, \ldots, a_r)\) in \( I_x \). By Nakayama’s lemma, this sequence generates \( I_x \). By hypothesis, the image of \((a_1, \ldots, a_r)\) in \( \Omega_{Z/k}(x) \) is linearly independent, and hence can be extended to a basis \( \omega(x) \) for \( \Omega_{Z/k}(x) \). Since \( Z/k \) is smooth, \( \Omega_{Z/k,x} \) is a free \( \mathcal{O}_{Z,x} \)-module, any lift \( \omega \) of \( \omega(x) \) to \( \Omega_{Z/k,x} \) will be a basis. It follows that the map \( I_x/I_x^2 \to i^* \Omega_{Z/k,x} \) is injective and locally split. The same holds in some neighborhood of \( x \), so (1) follows from Theorem 2. The proof that (1) implies (2) is immediate from this theorem.
Theorem 5 Let $X \to Z \to Y$ be morphisms of schemes. Assume that $X/Y$ and $Z/Y$ are smooth. Then $X/Z$ is smooth if and only if locally on $X$ the map $g^*\Omega_{Z/Y} \to \Omega_{X/Y}$ is injective and locally split.

Corollary 6 Let $X \to Y$ be a smooth morphism. Then, locally on $X$, there exists an étale factorization $X \to \mathbb{A}^n_Y \to Y$ of $X/Y$.

Proof: Let $x$ be a point of $X$. The image of $\mathcal{O}_{X,x} \to \Omega_{X/Y}(x)$ generates the $k(x)$-vector space $\Omega_{X/Y}(x)$, so there exists a sequence $(a_1, \ldots, a_n)$ in $\mathcal{O}_{X,x}$ whose image is a basis for $\Omega_{X/Y}(x)$. Get map $g: X \to \mathbb{A}^n_Y$ with $g^*t_i = a_i$ and $g^*dt_i = da_i$. Then $g^*\Omega_{\mathbb{A}^n_Y/Y} \to \Omega_{X/Y}$ is an isomorphism. By the previous result, $X/Y$ is smooth, and since $\Omega_{X/\mathbb{A}^n_Y} = 0$, it is also unramified. 

Example 7 Let $X$ be the closed subscheme of affine two space over $\mathbb{Z}[t]$ defined by $(x_1^3 + x_2^3 + 1 - 3tx_1x_2)$. Compute where $X/\mathbb{Z}$ is smooth and where $X/\text{Spec} \mathbb{Z}[t]$ is smooth. Do the same for the equation $t(x_1^4 + x_2^4 + 1) − 4x_1x_2x_3$, and for $t(x_1^4 + x_2^3 + 1) − 4x_1x_2x_3$. 

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