A Big Valuation Ring

Here is an example of a valuation ring of infinite Krull dimension. We shall use it to construct a quasi-affine scheme which has no closed point.

Recall that a monoid is a category with only one object, or equivalently, a set $M$ together with an associative binary operation with a two-sided identity element. Suppose that $M$ is commutative and cancellative, so that $ab = a'b$ implies that $a = a'$. Then $M$ can be embedded in an abelian group $G$, and it is clear that there is a smallest such group, unique up to unique isomorphism. We denote this by $M^{gp}$. If $x, y \in M^{gp}$, we write $x \leq y$ if $y = xz$ for some $z \in M$. This defines a partial preorder on $M$; it is a partial ordering if and only if $M$ has no units. Assume this is the case. We say that $M$ is valuative if the order it induces on $M^{gp}$ is a total order, equivalently, if for every $x \in M^{gp}$, either $x$ or $-x$ belongs to $M$. For example, the monoid of natural numbers under addition is valuative.

An ideal of a monoid $M$ is a subset $K$ such that $ak \in K$ if $a \in M$ and $k \in K$. An ideal is prime if $ab \in K$ implies $a$ or $b \in K$. Let $M^*$ be the set of units of $M$ and let $M^+ := M \setminus M^*$. This is maximal ideal of $M$ and it contains every proper ideal.

Lemma 1 Let $M$ be a valuative monoid, let $k$ be a field, and let $k[M]$ be the monoid algebra of $M$. (This is the free $k$-vector space with basis $M$ and with the evident structure of a $k$-algebra.) Then the subset $k[M^+]$ of $k[M]$ spanned by $M^+$ is a maximal ideal $P$ of $k[M]$, and the localization $k[M]_P$ is a valuation ring. The ideals of $k[M]_P$ are in natural bijection with the ideals of $M$.

Example 2 (thanks to G. Bergman) Let $G$ be the abelian group of polynomials with integer coefficients (under addition). Let $M$ be the submonoid consisting of those polynomials $p$ such that $p(t) \geq 0$ for all $t \in [0, \epsilon)$ for some $\epsilon > 0$. If $p(t) = a_0 + a_1 t + \cdots$, then $p(t) \in M$ if $p = 0$ or if the first nonzero $a_i$ (with smallest $i$) is positive. The corresponding order of $G$ is the lexicographical ordering, which is a total ordering. Thus $M$ is a valuative monoid. For each nonnegative integer $n$, The set $K_n$ of $p$ with $a_i > 0$ for some $i \leq n$ is a prime ideal of $M$, and we have

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_\infty.$$
where $K_\infty = \bigcup_n K_n$ is the maximal ideal of $Q$. Now let $V$ be the associated valuation ring and $S$ its spectrum. Then in $V$ we have a point $s_n$ corresponding to $K_n$ for $n = 0, 1, \ldots, \infty$, with $s_k$ a specialization of $s_j$ if and only if $k \geq j$. Let $X$ be the open subscheme of $S$ obtained by removing the closed point $s_\infty$. Then $X$ has no closed point.