A Big Valuation Ring

Here is an example of a valuation ring of infinite Krull dimension. We shall use it to construct a quasi-affine scheme which has no closed point.

Recall that a *monoid* is a category with only one object, or equivalently, a set M together with an associative binary operation with a two-sided identity element. Suppose that M is commutative and cancellative, so that $ab = a'b$ implies that $a = a'$. Then M can be embedded in an abelian group G, and it is clear that there is a smallest such group, unique up to unique isomorphism. We denote this by M^{gp}. If $x, y \in M^{gp}$, we write $x \leq y$ if $y = xz$ for some $z \in M$. This defines a partial preorder on M; it is a partial ordering if and only if M has no units. Assume this is the case. We say that M is valuative if the order it induces on M^{gp} is a total order, equivalently, if for every $x \in M^{gp}$, either x or $-x$ belongs to M. For example, the monoid of natural numbers under addition is valuative.

An ideal of a monoid M is a subset K such that $ak \in K$ if $a \in M$ and $k \in K$. An ideal is prime if $ab \in K$ implies a or $b \in K$. Let M^* be the set of units of M and let $M^+ := M \setminus M^*$. This is maximal ideal of M and it contains every proper ideal.

Lemma 1 *Let M be a valuative monoid, let k be a field, and let $k[M]$ be the monoid algebra of M. (This is the free k-vector space with basis M and with the evident structure of a k-algebra.) Then the subset $k[M^+]$ of $k[M]$ spanned by M^+ is a maximal ideal P of $k[M]$, and the localization $k[M]_P$ is a valuation ring. The ideals of $k[M]_P$ are in natural bijection with the ideals of M.*

Example 2 (thanks to G. Bergman) Let G be the abelian group of polynomials with integer coefficients (under addition). Let M be the submonoid consisting of those polynomials p such that $p(t) \geq 0$ for all $t \in [0, \epsilon)$ for some $\epsilon > 0$. If $p(t) = a_0 + a_1 t + \cdots$, then $p(t) \in M$ if $p = 0$ or if the first nonzero a_i (with smallest i) is positive. The corresponding order of G is the lexicographical ordering, which is a total ordering. Thus M is a valuative monoid. For each nonnegative integer n, The set K_n of p with $a_i > 0$ for some $i \leq n$ is a prime ideal of M, and we have

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots K_\infty,$$

where $K_\infty = \cup_n K_n$ is the maximal ideal of Q. Now let V be the associated valuation ring and S its spectrum. Then in V we have a point s_n corresponding to K_n for $n = 0, 1, \ldots, \infty$, with s_k a specialization of s_j if and only if $k \geq j$. Let X be the open subscheme of S obtained by removing the closed point s_∞. Then X has no closed point.