1.1 Important Examples

Exm 1: Euclidean space $\mathbb{R}^n = \{ \underline{x} = (x_1, \dots, x_n) : x_i \in \mathbb{R} \}$

norms:

Euclidean : $\| \underline{x} \| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

max : $\| \underline{x} \|_\infty = \max_j \{ x_j \}$

Rem: real dim-n vector space X is iso to $\mathbb{R}^n$.

Exm: $P_d = \{ f(x) = a_0 + a_1 x_1 + \dots + a_d x^d \}$. $\operatorname{Dim}(P_d) = d+1$,

so $P_d \cong \mathbb{R}^{d+1}$

other norm: $\| f \|_{L^1} := \int_0^1 | f(x) | \, dx$

Exm 2: $\mathbb{R}[x]$: dim $= \infty$, basis $= \{ 1, x, x^2, \dots \}$

norms : $\| f \|_{L^1} = \int_0^1 |f(x)| \, dx$

$\| f_\infty \| = \max_{\{j, a_j \neq 0\}} |a_j|$

Exm 3: $C[0,1]$ = continuous, real valued functions on $[0,1]$

dim $= \infty$, no natural basis.

norms : $\| f \|_{L^1} = \int_0^1 |f(x)| \, dx$

$\| f \|_{L^\infty} = \max_{x \in [0,1]} |f(x)|$

Exm 4: sequence space : $\ell^p := \{ \underline{x} = (x_j)_{j \geq 1} : \sum_{j=1}^\infty |x_j|^p < \infty \}$, $p \geq 1$.

norms: $\| \underline{x} \|_p = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p}$

$p = \infty$: $\ell^\infty := \{ \underline{x} = (x_j)_{j \geq 1} : \sup_{j \geq 1} |x_j| < \infty \}$

norm: $\| \underline{x} \|_\infty := \sup_{j \geq 1} |x_j|$

1.2. Normed Linear Spaces

Dfn 1: X v.s. over $\mathbb{F}$. *Norm* on X : $\| \cdot \| : X \to \mathbb{F}$ satisfying

- $\|x\| \geq 0 \ \forall x \in X$, $\|x\| = 0 \iff x = 0$;
- $\| \alpha x \| = |\alpha| \|x\| \ \forall x \in X, \alpha \in \mathbb{F}$
- $\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in X$

"normed linear space" $:= (X, \| \cdot \|)$.

Prop 2: $(X, \| \cdot \|)$. $d(x,y) := \|x - y\|$ gives metric space (X, d).

MORE EXAMPLES

Ⓐ $X = \mathbb{F}^n$, $p \geq 1$, set $\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p \right)^{1/p}$

Ⓑ X v.s over $\mathbb{F}$, dim $= n$. Every basis has exactly n elements

$\mathcal{B} = \{ e_1, \dots, e_n \}$, then $\forall x \in X$, $x = x_1 e_1 + \dots + x_n e_n$ (!).

norm: $\| \cdot \|_{p, \mathcal{B}} = \left(|x_1|^p + \dots + |x_n|^p \right)^{1/p}$

Ⓒ P_d, basis $= \{ 1, x, \dots, x^d \}$.

norms: $\| f \|_{L^p} := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$

: $\| f \|_{L^\infty} := \sup_{x \in [0,1]} |f(x)|$

Ⓓ $\mathbb{F}_0^\infty := \{ a = (a_1, a_2, a_3, \dots) : a_j \in \mathbb{F}$, finitely many $a_j \neq 0 \}$

$T : \mathbb{F}_0^\infty \to \mathbb{F}(x)$; $T : (a_1, \dots) \mapsto \sum_{j \geq 0} a_{j-1} x^j$

Ⓔ $\mathbb{F}^\infty := \{ x = (x_0, x_1, \dots) : x_j \in \mathbb{F} \}$

↳ sequence space ℓ^p is a subspace of $\mathbb{F}^\infty$.

Ⓕ $C[0,1]$: $\| f \|_{L^\infty} := \sup_{0 \leq x \leq 1} |f(x)|$

$\| f \|_{L^p} := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$

Ex 1: Reverse triangle inequality : $| \|x\| - \|y\| | \leq \|x + y\|$

Ex 2: $f(x) = \|x\|$ is continuous on X.

Ex 9: Each v.s. has a Hamel basis: $\{ e_a \}_{a \in A}$ s.t

$\forall x \in X$, x is a <u>finite</u> lin. comb. of some e_a's.

2.1 Inequalities

Lem 1: $A, B \geq 0$ and $0 \leq \theta \leq 1$. Then $A^{\theta} B^{1-\theta} \leq \theta A + (1-\theta) B$

Notation: $1 < p < \infty$, set $q = \frac{p}{(p-1)}$ so that $\frac{1}{p} + \frac{1}{q} = 1$

Thm 2: (Hölder's Inequality) for any two sequences $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$, $a_i, b_i \geq 0$, for any $1 < p < \infty$:

- $$\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^p \right)^{1/p} \cdot \left(\sum_{j=1}^{n} b_j^q \right)^{1/q} \quad (H)$$

- when $p = 1$, $\sum_{j=1}^{n} a_j b_j \leq \sum_{j=1}^{n} a_j \cdot \max_{1 \leq j \leq n} b_j$ (H')
- $p = \infty$, $\sum_{j=1}^{n} a_j b_j \leq \max_{1 \leq j \leq n} a_j \cdot \sum_{j=1}^{n} b_j$ (H'')

Cor 3: $x = (x_j)_{j \geq 1}$, $y = (y_j)_{j \geq 1}$ complex seq. $1 < p < \infty$:

$$\left| \sum_{j=1}^{\infty} x_j y_j \right| \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^q \right)^{1/q} \quad (H\text{-complex})$$

Thm 4 (Minkowski's Inequality): $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of $\mathbb{F}^n$, then

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} \quad (M)$$

or

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \quad \text{for } 1 < p < \infty$$

$n \to \infty$ extends this to infinite sequences.

Prop 5: (Minkowski's Inequality for integrals)
$f, g \in C[0,1]$, (M - Int)

$$\left(\int_0^1 |f(x) + g(x)|^p \, dx \right)^{1/p} \leq \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} + \left(\int_0^1 |g(x)|^p \, dx \right)^{1/p}$$

2.2 Metric Structure of Normed Linear Spaces.

Dfn 1: $G \subset X$ open if $\forall x \in G$, $\exists r > 0$ s.t. $B_r(x) \subseteq G$
$G \subset X$ closed if $X \setminus G$ is open.

Dfn: Subspace is a subset $Y \subseteq X$ where every linear combination of elements in Y is contained in Y.
I.e. $x, y \in Y \Rightarrow \alpha x + \beta y \in Y \quad \forall \alpha, \beta \in \mathbb{F}$.

Rem: if Y a subspace, then $0 \in Y$.

Prop 2: $Y \subseteq X$ a subspace. If Y open, then $Y = X$.

Dfn: Closure of $Y = \overline{Y}$:= set of limit points of S
:= smallest closed set containing S.

Lem 3: Y lin subspace. Then $\overline{Y}$ is also a linear subspace.

Dfn 4: Y is dense in X if $\overline{Y} = X$.

Dfn 5: $K \subseteq (X, d)$ is compact if every open covering of K has a finite subcover.

Prop 6: $K \subseteq (X, d)$ is compact iff every infinite sequence $\{x_n\}$ in K has a subseq. which converges to a point in K
Compact $\Leftrightarrow$ sequentially compact

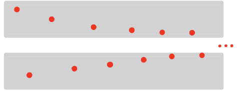
Dfn 7: set $E \subseteq (X, \|\cdot\|)$ is bounded if $\exists r > 0$ s.t. $E \subset B_r(0)$.

Thm 9: $K \subseteq (\mathbb{F}^n, \|\cdot\|_p)$ is compact $\Leftrightarrow$ K is closed and bounded.

Cor 10: X fin. dim v.s. with basis B. For any $1 \leq p \leq \infty$, a set K in $(X, \|\cdot\|_{p,B})$ is compact iff K is closed and bounded.

Rem: $(\ell^p, \|\cdot\|_q)$ is an NLS for $q \geq p > 1$.
But $p < q \not\Rightarrow$ not NLS.

3.1. Banach Spaces



"Holes"

Metric space is **complete** if every Cauchy sequence $\{x_n\} \subset X$ converges to a point in X.

Dfn 1: NLS $(X, \|\cdot\|)$ is called a **Banach space** if the associated metric space (X, d) is a complete metric space.

EXAMPLES

Ⓐ Finite dimensional NLS

Ⓑ $(C[0,1], \|\cdot\|_{L^\infty})$: $d(x,y) = \sup_{x,y \in [0,1]} |f(x) - f(y)|$

Ⓒ NOT Banach spaces: $(C[0,1], \|\cdot\|_{L^p})$, $1 \le p < \infty$.

Counterexample: $\{f_n = x^n\}$

Ⓓ $(\ell^p, \|\cdot\|_p)$, $1 \le p \le \infty$

Lem 2: $(E, \|\cdot\|)$ a Banach space and let Y be a linear subspace of E. $(Y, \|\cdot\|)$ is a Banach space iff Y is a closed linear subspace of E

Completion of NLS: $(X, \|\cdot\|) \xrightarrow{\text{complete}} (E, \|\cdot\|')$, where E complete, X dense in E, and $\|x\| = \|x\|'$ $\forall x \in X$.

① $C := \{x = (x_j)_{j \ge 1} : x \text{ cauchy in } X\}$. Equiv. rel. $\sim$ on C: $x \sim y \Leftrightarrow \|x_j - y_j\| \to 0$ as $j \to \infty$. $E = C/\sim$.
E is a v.s: $[x] + [y] := [x+y]$, $\alpha[x] := [\alpha x]$.

② $\forall x \in C$, $\big| \|x_j\| - \|x_k\| \big| \le \|x_j - x_k\| \to 0$ as $j \to \infty$
$\Rightarrow \{\|x_j\|\}$ is cauchy in $\mathbb{R}$
$\mathbb{R}$ complete $\Rightarrow \lim_{j\to\infty} \|x_j\|$ exists.
$\|\cdot\|' := \|[x]\|' := \lim_{j\to\infty} \|x_j\|$

③ Identify $x \in X$ with $[x'] = [\{x, x, x, \dots\}]$.

④ $(E, \|\cdot\|')$ is a Banach space.

Dfn 3: $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS. Then:

- Linear map $T: X \to Y$ is called an *isometry* if $\|Tx\|_Y = \|x\|_X$ $\forall x \in X$
- X and Y are called *isometrically isomorphic* if $\exists$ an isometry $T: X$ onto Y. Then $T^{-1}: Y \to X$ is automatically a surjective isometry (T linear).
- Banach space completion of $(X, \|\cdot\|_X)$ is a banach space $(Y, \|\cdot\|_Y)$ and an isometry $T: X \to Y$ s.t $T(X)$ is dense in Y.

Prop 4: $\{T, (Y, \|\cdot\|_Y)\}$ and $\{S, (Z, \|\cdot\|_Z)\}$ two completions of $(X, \|\cdot\|_X)$. Then Y and Z are isometrically isomorphic.

3.2 Equivalence of Norms

Dfn 1: Two norms $\|\cdot\|_1, \|\cdot\|_2$ are *equivalent* if $\exists$ a constant A s.t $\forall x \in X$,
$\|x\|_1 \le A\|x\|_2$ and $\|x\|_2 \le A\|x\|_1$.
$\Leftrightarrow$ $B_1^1(0) \subseteq A B_1^2(0)$, $B_1^2(0) \subseteq A B_1^1(0)$

Rem: can also show $\exists A, B \in \mathbb{F}$ s.t $\forall x \in X$,
$\|x\|_1 \le A\|x\|_2$, and $\|x\|_2 \le B\|x\|_1$.

Rem: equivalent norms is an equivalence relation

Prop 2: $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on X:

- If $\{x_n\} \to x$ wrt. $\|\cdot\|_1$, then $\{x_n\} \to x$ wrt $\|\cdot\|_2$.
- If $G \subset X$ open wrt $\|\cdot\|_1$, then G open wrt $\|\cdot\|_2$
- $(X, \|\cdot\|_1)$ is Banach $\Leftrightarrow$ $(X, \|\cdot\|_2)$ is Banach.

Thm 3: Let X be a finite dim v.s over $\mathbb{F}$. Then all norms on X are equivalent.

Cor 4: $(X, \|\cdot\|)$ fin dim NLS. Then $(X, \|\cdot\|)$ is a Banach space. If Y is a linear subspace of X, then Y is closed.

Cor 5: $(X, \|\cdot\|)$ fin. dim NLS. Then a subset $K \subset X$ is compact iff K is closed and bounded.

Thm 6: $(X, \|\cdot\|)$ NLS. $\overline{B_1(0)}$ is compact iff X is finite dimensional.

4.1 Inner Product Spaces

Dfn 1: X = V.S. over $\mathbb{F}$. An *inner product* on X is a map $\langle \cdot,\cdot \rangle : X \times X \to \mathbb{F}$ satisfying

① $\langle x,x\rangle \geq 0 \quad \forall x \in X$, $\langle x,x\rangle = 0 \Leftrightarrow x = 0$

② $\langle x,y\rangle = \overline{\langle y,x\rangle} \quad \forall x,y \in X$

③ $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$ $\quad \forall x,y,z \in X$, $\forall \alpha,\beta \in \mathbb{F}$

Note: $\langle x, \alpha y\rangle = \bar{\alpha}\langle x,y\rangle$ but $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$.

norm: $\|x\| := \sqrt{\langle x,x\rangle}$

EXAMPLES

(A) $\mathbb{F}^n$, $\langle x,y\rangle = x_1\bar{y}_1 + \dots + x_n\bar{y}_n$

$\sqrt{\langle x,x\rangle} = \|x\|_2$

(B) ℓ^2, $\langle x,y\rangle = \sum_{j=1}^{\infty} x_j \bar{y}_j$

$\sqrt{\langle x,x\rangle} = \|x\|_2$

(C) $C[0,1]$, $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

$\sqrt{\langle x,x\rangle} = \|f\|_{L^2}$.

Thm 2 (Cauchy-Schwarz Inequality)

$(X, \langle\cdot,\cdot\rangle)$. $\forall x \in X$,

$|\langle x,y\rangle| \leq \|x\|\,\|y\|$

Useful fact: $\operatorname{Re}\langle x,y\rangle \leq |\langle x,y\rangle| \leq \|x\|\,\|y\|$.

Dfn: x and y are *orthogonal* if $\langle x,y\rangle = 0$.

notation: $x \perp y$.

Prop 4 (Pythagoras) If $x \perp y$, then (IPS)

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Prop 5 (Parallelogram law) $\forall x,y \in X$, (IPS)

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Rem: if X an (real) NLS + has parallelogram law, then can construct norm: $4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$.

Prop 6: $(X, \langle\cdot,\cdot\rangle)$ with scalars $\mathbb{C}$. Then

$4\langle x,y\rangle = \sum_{k=0}^{3} i^k \|x + i^k y\|^2$

$= \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

So called "polarization identities of $\mathbb{R}$ and $\mathbb{C}$".

4.2. Orthogonality

orthogonal projection

$\operatorname{Span}(y) := \{\alpha y : \alpha \in \mathbb{F}\}$

$P_y(x) := \langle x,y\rangle y$ ← linear map, $\subset \operatorname{Span}(y)$.

$P_y(x) \perp (x - P_y(x))$, $\|x - P_y(x)\| = \inf\{\|x - \alpha y\| : \alpha \in \mathbb{F}\}$

Y fin. dim subspace of IPS $(X, \langle\cdot,\cdot\rangle)$, then Y has an *orthonormal basis* (ONB): $Y = \operatorname{span}(e_1,\dots,e_n)$, $\langle e_i, e_j\rangle = \delta_j^i$

$y = \sum_{j=1}^{n} \alpha_j e_j \Rightarrow \|y\|^2 = \sum_{j=1}^{n} |\alpha_j|^2$

Thm 1: $Y \subset (X, \langle\cdot,\cdot\rangle)$ IPS, ONB of Y $\{e_j\}_{j=1}^n$. The linear map $P_Y : X \to Y$ given by $P_Y x = \sum_{j=1}^{n} \langle x, e_j\rangle e_j$ satisfies:

① $(x - P_Y x) \perp y \quad \forall y \in Y$

② Bessel's Inequality: $\sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \leq \|x\|^2$

equivalently: $\|P_Y x\|^2 \leq \|x\|^2$.

③ $P_Y x$ is the unique closest point in Y to x

$\|x - P_Y x\| = \min\{\|x - y\| : y \in Y\}$.

EXAMPLES:

(C) Use Thm 1 to compute $\inf_{\substack{a_0,a_1,\dots,\\ a_{n-1} \in \mathbb{R}}} \int_0^1 |x^n - a_{n-1}x^{n-1} - \dots - a_1 x - a_0|^2\,dx$

Method:

1] Consider $n+1$ dim IPS $(P_n, \|\cdot\|_{L^2})$, $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

let $Y = \{p(x) = \sum_{j=1}^{n-1} a_j x^j\}$

2] Construct ONB for Y using ① of Thm 1.

3] By Thm 1, the poly $p \in Y$ that achieves the min in the integral is $P_Y f(x) = \sum_{j=0}^{n-1} \langle f, e_j\rangle e_j(x)$, where $f(x) = x^n$.

Gram-Schmidt Process (make ONB)

$\operatorname{proj}_u(v) := \dfrac{\langle u,v\rangle}{\langle u,u\rangle} u$

$u_1 = v_1$

$u_2 = v_2 - \operatorname{proj}_{u_1}(v_2)$

$u_3 = v_3 - \operatorname{proj}_{u_1}(v_3) - \operatorname{proj}_{u_2}(v_3)$

$u_4 = v_4 - \operatorname{proj}_{u_1}(v_4) - \operatorname{proj}_{u_2}(v_3) - \operatorname{proj}_{u_3}(v_4)$

$\vdots$

$u_k = v_k - \sum_{j=1}^{k-1} \operatorname{proj}_{u_j}(v_k)$

$$\boxed{e_k = \frac{u_k}{\|u_k\|}}$$

5.1. Hilbert Spaces

Dfn: a Hilbert space is an IPS $(X, \langle\cdot,\cdot\rangle)$ where the metric induced by $\langle\cdot,\cdot\rangle$ is complete.

Hilbert spaces $\subset$ Banach spaces.
$\downarrow$
Banach + IPS

EXAMPLES:

Ⓐ $\mathbb{F}^n$, $\langle x,y\rangle = \sum x_i \bar{y}_i$
Ⓔ ℓ^2, $\langle x,y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$
Ⓕ $C[0,1]$, $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

Parseval's Identity: Infinite dim Hilbert space, $\{e_1, e_2, \dots\}$ orthonormal sequence of vectors. Then

$$\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2$$

Every element $x \in H$ has a unique expansion as an infinite linear combination of $\{e_j\}_{j\geq 1}$.

Dfn: $C \subset V$ over $\mathbb{F}$ is called convex if whenever $x, y \in C$, then the line segment $\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\} \subset C$.

Rem: any linear subspace is convex.

Thm 1: $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space and $C \subset H$ a closed, convex set. Then $\forall x \notin C$, $\exists!\ y_0 \in C$ s.t

$$\|x - y_0\| = \inf\{\|x-y\| : y \in C\}$$

Dfn 2: Let S be any set of vectors in an IPS. The orthogonal complement to S is

$$S^\perp := \{x \in X : x \perp y \quad \forall y \in S\}$$

$S^\perp$ = linear, closed, $S \cap S^\perp = 0$.

Thm 3: H Hilbert, M proper, closed, linear subspace of H. $\exists$ linear maps $P_M : H \to M$, $Q_M : H \to M^\perp$ s.t $\forall x \in H$,

$$x = P_M x + Q_M x$$

This representation is unique, and P_M is called the orthogonal projection of H onto M.

$$H = M \oplus M^\perp$$

5.2. Orthonormal Bases

Notation: Fourier coefficients: $\hat{x}(j) := \langle x, e_j\rangle$.

Thm 1: CONDITIONS FOR ORTHONORMAL BASIS

$\Leftrightarrow$ $\forall x \in H$, if $\hat{x}(j) = 0 \ \forall j \geq 1$, then $x = 0$. I.e. the sequence $\{e_j\}_{j\geq 1}$ is a maximal orthonormal family of vectors. (only 0 is ortho to all e_j)

$\Leftrightarrow$ $H = \overline{\text{span}(\{e_j\}_{j\geq 1})}$

$\Leftrightarrow$ $\forall x \in H$, $x = \sum_{j=1}^{\infty} \hat{x}(j) e_j = \lim_{n\to\infty} \sum_{j=1}^{n} \hat{x}(j) e_j$

$\Leftrightarrow$ $\forall x, y \in H$, $\langle x, y\rangle = \sum_{j=1}^{\infty} \hat{x}(j)\overline{\hat{y}(j)}$

$\Leftrightarrow$ $\forall x \in H$, Parseval's identity holds:

$$\|x\|^2 = \sum_{j=1}^{\infty} |\hat{x}(j)|^2$$

One of these satisfied $\Rightarrow$ $\{e_j\}$ an ONB for H.

Rem: $\dim(H)$ finite, then $(H, \langle\cdot,\cdot\rangle)$ and $(\mathbb{F}^n, \langle\cdot,\cdot\rangle_2)$ are indistinguishable.

$$\boxed{\langle x, y\rangle = \langle (\hat{x}(j))_{j=1}^n, (\hat{y}(j))_{j=1}^n \rangle_2}$$

Define: $(\dim(H) = \infty)$ $T : H \to \ell^2$; $Tx = (\hat{x}(j))_{j\geq 1}$.
↳ T is an isometry

$$\langle x, y\rangle = \langle (\hat{x}(j))_{j\geq 1}, (\hat{y}(j))_{j\geq 1} \rangle_{\ell^2}$$

Prop 2: T is onto

Rem: $\dim(H) = \infty$, has ONB, then $(H, \langle\cdot,\cdot\rangle)$ and $(\ell^2, \langle\cdot,\cdot\rangle_{\ell^2})$ are indistinguishable.

EXAMPLES:

① $(\ell^2, \|\cdot\|_{\ell^2})$: ONB = $\{e_j\}$, $e_j = (0, 0, \dots, 0, \overset{j\text{th entry}}{1}, 0, 0, \dots 0)$

② $(C[0,1], \langle\cdot,\cdot\rangle)$, $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$.
$\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ forms ONB for Completion of $C[0,1]$, $L^2[0,1]$.

Prop 3: H possesses an ONB iff $\exists$ a countable set $S \subset H$ of vectors which is dense in H

Has ONB $\rightarrow$ called a separable Hilbert space.

6.1. Bounded Linear Operators I.

Linear map: $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$

Continuous: $x_n \to x$, then $T(x_n) \to T(x)$.

Dfn 1: $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ NLS, and $T: X \to Y$ a linear map. T is a *bounded linear operator* (BLO) if $\exists\, M \geq 0$ s.t

$$\|Tx\| \leq M\|x\|$$

$\forall x \in X$.

may have different norms on X and Y

Ex: $\forall x \in X$, $\|Tx\| \leq M\|x\|$

$\Leftrightarrow$ $\|x\| \leq 1$, $\|Tx\| \leq M$

$\Leftrightarrow$ $\|x\| = 1$, $\|Tx\| = M$

$\Leftrightarrow$ $\forall x \neq 0$, $\frac{\|Tx\|}{\|x\|} = M$

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \leq 1} \|Tx\|$$

and

$$\|Tx\| \leq \|T\|\|x\| \quad \forall x \in X$$

Thm 2: T is continuous at 0: $x_n \to 0 \Rightarrow Tx_n \to T0 = 0$.

$\Leftrightarrow$ T is continuous at every point $x \in X$: $x_n \to x$, $Tx_n \to Tx$

$\Leftrightarrow$ T is bounded

Ex: $\|T\| = \min\{M : \|Tx\| \leq M\|x\| \;\; \forall x \in X\}$

Prop: operator norm $\|T\|$ on BLS: $\mathcal{L}(X,Y)$ is a norm

EXAMPLES:

(A) Shift operators: LSO: $L: \ell^p \to \ell^p$, RSO: $R: \ell^p \to \ell^p$,

$Lx = (x_2, x_3, \dots)$, $Rx = (0, x_1, x_2, x_3, \dots)$

$\|Rx\|_p = \|x\|_p$, $\|Lx\|_p \leq \|x\|_p$.

(B) $(C[0,1], \|\cdot\|_{L^1})$. $T: C[0,1] \to \mathbb{F}$; $Tf = \int_0^1 x f(x)\,dx$

$\Rightarrow |Tf| \leq \|f\|_{L^1}$

Prop 4: $T: X \to Y$ lin. op, $\dim(X) < \infty$. Then T is continuous.

Eval/vec: $Tx = \lambda x \Rightarrow \sup_{\text{evals } \lambda} |\lambda| \leq \|T\|$

Spectral Radius: $\rho(T) := \sup|\lambda|$

↳ independent of norm.

Self adjoint: $\langle Tx, y\rangle = \langle x, Ty\rangle \Rightarrow \rho(T) = \|T\|$

↳ all real evals, basis of evecs.

Adjoint: T^* of $T :=\langle Tx, y\rangle = \langle x, T^* y\rangle$

T^*T = self adjoint.

Singular values of T: = evals of T^*T.

6.2. Bounded Linear Operators

Thm 1: Y Banach $\Rightarrow \mathcal{L}(X,Y)$ Banach

Prop 2: (Extension from a dense subspace)

X NLS, Y Banach, $X' \subset X$ dense, linear subspace. Suppose $T \in \mathcal{L}(X', Y)$. Then $\exists!\, \tilde{T} \in \mathcal{L}(X,Y)$ s.t $\tilde{T}|_{X'} = T$, $\|\tilde{T}\| = \|T\|$, and T an isometry $\Rightarrow \tilde{T}$ an isometry.

Prop 3: Let $T \in \mathcal{L}(X,Y)$. $\mathrm{Ker}(T)$ is a closed subspace.

Finite Rank Operators (FRO)

Finite Rank operator := $FR(X,Y) = \{T \in \mathcal{L}(X,Y) : \dim(\mathrm{Im}(T)) < \infty\}$

Prop 4: $T \in \mathcal{L}(X,Y)$, $T(X)$ fin dim, $c_j : X \to \mathbb{F}$, $1 \leq j \leq m$ lin. coord. maps of T wrt. a basis $\mathcal{B} = \{f_j\}_{j=1}^m$ of $T(X)$.

Then T is continuous iff c_j is continuous $\forall\, 1 \leq j \leq m$.

$X^* := \mathcal{L}(X, \mathbb{F})$ = space of *linear functionals*.

7.1 Dual spaces

Dfn 1: X NLS over $\mathbb{F}$. The *dual space* to X is the space $\mathcal{L}(X,\mathbb{F}) :=$ space of all cont, lin maps $X \to \mathbb{F}$.
Notation: $X^* := \mathcal{L}(X,\mathbb{F})$, and $T \in X^*$ is called a *bounded linear functional* on X.

X^* is complete always.

Prop 2: Let $1<p<\infty$. The dual of ℓ^p is isometrically isomorphic to ℓ^q, where $1/p + 1/q = 1$.

Sgn: $\mathbb{C} \to \{-1, 0, +1\}$

Ex:
- $(\ell^1)^*$ iso iso to ℓ^∞
- $(c_0)^*$ iso iso to ℓ^1
- isometric embedding of ℓ^1 into (ℓ^∞)

Thm 3: (Riesz Representation Theorem)
Let H be a complex Hilbert space. Then $\exists$ a conjugate-linear isometric isomorphism from H onto H^* given by $T: y \mapsto T_y$, where $T_y x = \langle x, y\rangle$.

Rem: $\|T_y\| = \|y\|$.

Rem: $\mathbb{F} = \mathbb{R} \Rightarrow T$ is linear.

7.2. Adjoints

matrix of $T = (a_{ij})$, then matrix of T^* is $(\overline{a_{ji}})$

Thm 1: H, K Hilbert, $T \in \mathcal{L}(H,K)$. $\forall y \in K$, $\exists!\, T^*y \in H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x \in H$
The map $y \mapsto T^*y$ is $T^* \in \mathcal{L}(K,H)$, satisfying $\|T^*\| = \|T\|$.

Rem: $T^{**} = T$

Prop 2: H, K Hilbert, $T \in \mathcal{L}(H,K)$. Then
$$\|T^*T\| = \|TT^*\| = \|T\|^2 = \|T^*\|^2$$

Integral operators

$T: C[0,1] \to C[0,1]$ linear: $Tf(x) = \int_0^1 k(x,y) f(y)\,dy$,
where $k \in C([0,1] \times [0,1])$. $k :=$ *kernel* of T
k = example of an *integral operator.*

Can extend to Hilbert space

Then if T is an integral operator with kernel $k(x,y)$, then its adjoint T^* is also an integral operator, with $k^*(x,y) = \overline{k(y,x)}$

↳ e.g. $L^2[0,1]$ is completion of $C[0,1]$.

8.1 Transposes

Extending idea of adjoints to all normed spaces, not just Hilbert.

Dfn 1: Let X, Y NLS and $T \in \mathcal{L}(X,Y)$. The *transpose of T*, denoted by T', is the map $T': Y^* \to X^*$ given by $(T'g)(x) = g(T(x))$

Prop 2: X, Y NLS and $T \in \mathcal{L}(X,Y)$. Then $T': Y^* \to X^*$ is a BLO and $\|T'\| \leq \|T\|$.

Prop 3: X, Y NLS and $T \in \mathcal{L}(X,Y)$. If Y satisfies $\|y\| = \sup_{g \in Y^*, \|g\| \leq 1} |g(y)|$ $\forall\, y \in Y$, then $\|T'\| = \|T\|$.

Reflexivity: X normed space, $x \in X$, define action of x on X^* by $\tilde{x} : X^* \to \mathbb{F}$; $\tilde{x} : f \mapsto f(x)$.

Then $\tilde{x} \in X^{**}$. Define $\iota : X \to X^{**}$; $\iota : x \mapsto \tilde{x}$. Linearity implies ι is an isometry of X onto X^{**}

Dfn: Banach space X is *reflexive* if $\iota : X \to X^{**}$ is surjective.

Ex: any Hilbert space is reflexive.

Rem: X, Y reflexive, $T \in \mathcal{L}(X,Y)$. Then $T'' = \iota_Y \circ T \circ \iota_X^{-1}$.

8.2. Hilbert - Schmidt Operators.

H, K separable Hilbert (both have ONBs)

Dfn 1: A BLO $T: H \to K$ is called a *Hilbert - Schmidt operator* if

$$\|T\|_{HS}^2 := \sum_n \|Te_n\|^2 < \infty$$

where $\{e_n\}$ is any ONB for H.

$\mathcal{HS}(H,K) :=$ set of Hilbert - Schmidt operators between H and K.

Rem:
- $\mathcal{HS}(H,K)$ is a Lin. sub. of $\mathcal{L}(H,K)$
- $\|T\|_{HS}$ is a norm on $\mathcal{HS}(H,K)$.
- $\dim(H) < \infty$, then $\mathcal{HS}(H,K) = \mathcal{L}(H,K)$.
- $FR(H,K) \subset \mathcal{HS}(H,K)$.

Lem 2: Let $\{e_n\}$ be an ONB for H, $\{f_m\}$ an ONB for K.
Then
$$\sum_n \|Te_n\|^2 = \sum_{n \geq 1} \sum_{m \geq 1} |\langle Te_n, f_m \rangle|^2 = \sum_m \|T^* f_m\|^2$$

Prop 3: H, K two separable Hilbert spaces, $T \in \mathcal{HS}(H,K)$.
Then $\|T\| \leq \|T\|_{HS}$

Prop 4: Let H, K separable, Hilbert, $T \in \mathcal{HS}(H,K)$
Then $\exists$ sequence $\{T_N\} \subseteq FR(H,K)$ s.t $T_N \to T$ in $\mathcal{L}(H,K)$: $\|T_N - T\| \to 0$ as $N \to \infty$.

9.1 Compact Operators

Dfn 1: X, Y NS, $T: X \to Y$ linear map. Then T is called *compact* if $\overline{T(\{x : \|x\| \leq 1\})}$ is a compact subset of Y.

Rem: any compact operator is a bounded operator.

$K(X,Y) = \{$ compact operators $X \to Y\}$.

Rem:
- $K(X,Y) \subseteq \mathcal{L}(X,Y)$
- $FR(X,Y) \subseteq K(X,Y)$

Thm 2: X NS, Y Banach, and $T_j \in \mathcal{L}(X,Y)$ a compact operator. Suppose $\exists\; T \in \mathcal{L}(X,Y)$ s.t $\|T_j - T\| \to 0$ as $j \to \infty$. Then T is compact.

Lem 3: X, Y NS, Suppose $T \in \mathcal{L}(X,Y)$. Then T is compact iff $\forall$ bounded sequence $\{x_n\} \subseteq X$, the sequence $\{Tx_n\} \subseteq Y$ has a convergent subsequence.

Cor 4: X, Y NS. Then $K(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$.

9.2. Compact Operators II

Hilbert Spaces: $FR \subseteq \mathcal{HS} \subseteq K \subseteq \mathcal{L}$

$T \in \mathcal{HS}(H,K) \Leftrightarrow T^* \in \mathcal{HS}(K,H)$, and $\|T\|_{HS} = \|T^*\|_{HS}$

Prop 1: H, K separable, Hilbert, and $T: H \to K$ linear. If $\{e_n\}$ is ONB of H s.t $\sum_n \|Te_n\|^2 < \infty$, then $T \in \mathcal{L}(H,K)$ and hence $T \in \mathcal{HS}(H,K)$.

10.1 Spectral Theorem – Preliminaries

The Theorem...

Thm 1: H sep, Hilbert, and $T \in \mathcal{L}(H)$ is a compact, self-adjoint operator. Then $\exists$ an orthonormal basis of eigenvectors $\{e_n\}$ for T so that $\forall x \in H$,

$$Tx = \sum_n \lambda_n \langle x, e_n \rangle e_n$$

Moreover, the eigenvalues λ_n are real and when $\dim H = \infty$, the sequence $\{\lambda_n\}$ belongs to c_0. Finally, the eigenspace of each nonzero eigenvalue is infinite dimensional.

Cor 2: If $T \in \mathcal{L}(H)$ is a compact, self-adjoint operator, then $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue for } T\}$.

Lem 3: $T \in \mathcal{L}(H)$ self-adjoint, and for some $0 \neq x \in H$, $Tx = \lambda x$. Then $\lambda \in \mathbb{R}$.

Lem 4: $T \in \mathcal{L}(H)$ self-adjoint, and for some $0 \neq x \in H$, $Tx = \lambda x$ and for some $0 \neq y \in H$, $Ty = \mu y$, $\lambda \neq \mu$. Then $\langle x, y \rangle = 0$.

Prop 5: $T \in \mathcal{L}(H)$ compact, self-adjoint. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue for T.

Lem 6: $T \in \mathcal{L}(H)$ self-adj. Then $\|T\| = \sup_{\|x\| \leq 1} |\langle Tx, x \rangle|$

Rem: $T \in \mathcal{L}(H)$ self-adj, then $\langle Tx, x \rangle \in \mathbb{R}$ $\forall x \in H$.

10.2 Spectral Theorem, the Proof

Rem: projection $P: V \to V$ is self-adj iff P is orthogonal.

Rem: Orthogonal projection

A Spectral Theorem for Normal Operators

Normal: $TT^* = T^*T$

Thm 1: Let H be a seperable (complex) Hilbert space and suppose $T \in \mathcal{L}(H)$ is a compact normal operator. Then $\exists$ an ONB of eigenvectors $\{e_n\}$ for T s.t $\forall x \in H$,

$$Tx = \sum_n \lambda_n \langle x, e_n \rangle e_n$$

Moreover, the sequence $(\lambda_n) \in c_0$. Finally, the eigenspace of each nonzero eigenvalue is finite-dimensional.