Dehn Surgery and Non-Separating 2-Spheres

Nancy Mae Eagles

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1 Introduction

Set up:
Let Y be a closed, orientable 3-manifold, and let $K : S^1 \hookrightarrow Y$ be a knot. In knot theory and 3-manifold theory, there are two major types of problems: *geographical*, which is concerned with which manifolds may be constructed, and *botanical*, which is concerned with how one could construct a fixed manifold from another. I.e., where can you go, and how you can get there.

Question. What manifolds can be created by doing surgery on a knot in S^3? More precisely, what is $Y_n(U)$ or $Y_n(K)$?

Question. What surgeries give us a certain manifold? More precisely, what are the restraints on n and k such that $Y_n(K) \simeq M$ for a particular M?

Remark 1. $Y = S^3$, $K = U$ and $n = 0$, we get $S^2 \times S^1$.

Proof. Recall the procedure of doing surgery. Identify a canonical seifert longitude and a meridian of a tubular neighborhood of k. Such a tubular neighborhood has boundary homeomorphic to $S^1 \times S^1$. We may remove the neighborhood and glue it back in by an automorphism of the boundary, which homologically is governed by a choice of linear combination of l and m to map m to: $m \mapsto pm + ql$. In the case of $p/q = 0$, we identify m with Seifert longitude, which bounds a disk in the complement of $N(K)$. The result is $(D^2 \cup_{S^1} D^2) \times S^1 = S^2 \times S^1$. □

Question. What about other K, n? Precisely, when can surgery on a knot in S^3 produce $S^2 \times S^1$?

Theorem 1.1 (Gabai, '87)**.** $S^3_0(K) \simeq S^2 \times S^1$ if and only if K is the unknot.

Remark 2. What about $S^3_n(U)$? These are the lens spaces $L(n, 1)$. These are definitively not $S^2 \times S^1$, since they have different homology groups. And in the case that we are dealing with $S^3_n(K)$ where $n \neq 0$ and $K \neq U$, Kirby calculus tells you that it can be expressed as integral surgery along a bunch of (possibly linked) unknots, which I suppose is going to probably differ from $S_0(U)$ anyway.

So, how do we generalize this further? The proof of Gabai's Property R theorem theorem uses taut foliations. However, this was very particular to the case of S^3, and so not generalizable in that direction.

Remark 3. An orientable 3-manifold M contains a nonseparating S^2 if and only if it has an $S^2 \times S^1$ summand.

Proof. Let's suppose first that M is prime, i.e. cannot be written as a nontrivial connect sum. Then certainly if $M = S^2 \times S^1$, then $S^2 \times \{p\}$ is a nonseparating 2-sphere. Indeed, the complement is $S^2 \times I$, which is connected. On the other hand, suppose that M contains a nonseparating S^2. The idea is to use this nonseparatingness to build up a tubular neighborhood of S^2 into a copy of $S^2 \times S^1$, and then see the remainder of the manifold as a connect sum with this space.

Lemma 1.2. $S^2 \hookrightarrow M$ is a nonseparating 2-sphere if and only if there exists a loop $\ell \subset M$ that intersects S^2 exactly once.

Proof. Suppose $S^2 \hookrightarrow M$ is nonseparating. Take a small arc intersecting S^2 transversely. Then the end points lie in the complement of S^2 in M. Since $M\S^2$ is connected, we can connect the two end points of M with an arc, closing up the first arc to make a loop. Now suppose that S^2 is separating. Then S^2 divides M into two connected components. Any loop intersecting S^2 transversely has to alternate between these components (e.g. interior to exterior to interior), and so for the sake of closing up must intersect the sphere an even number of times. □

With this lemma established, take a tubular neighborhood of $S^2 \hookrightarrow M$, $S^2 \times (-\epsilon, \epsilon)$. Since S^2 is non-separating, we can find a loop that intersects S^2 transversely, and exactly once. Take a tubular neighborhood of the part of this loop outside S^2's tubular neighborhood. Adjusting its width if necessary, we can arrange that this $D^2 \times I$ meets $S^2 \times (-\epsilon, \epsilon)$ in two small disks in $S^2 \times \{\pm\epsilon\}$.

Define $X := (S62 \times (-\epsilon, \epsilon) \cup (D^2 \times I)$. Note that ∂X is homeomorphic to S^2, since $\partial X = (S^2 \setminus D^2) \times \{-\epsilon\} \cup (\partial D^2 \times [0, 1]) \cup (S^2 \setminus D^2) \times \{\epsilon\}$. Now, since M is a connected, compact, oriented 3-manifold, $M - X$ and X are also compact. We may express

$$M = X \cup_{\partial X} (M \setminus X).$$

In other words, we may express M as a connect sum. Since M is prime, it must be that $M \setminus X$ is B^3. If you keep track of the gluing maps through this, I think you end up gluing in a 3-ball that completes the $D^2 \times I$ part to an $S^2 \times I$ part, giving $S^2 \times S^1$. □

In 2020, Hom and Lidman generalized Gabai's Property R theorem to rational homology spheres:

Theorem 1.3 (Generalized Property R Theorem for $\mathbb{Q}HS^3$)**.** If 0-framed surgery on a nullhomotopic knot $K \subset Y$ has a non-separating 2-sphere, then K is the unknot.

In other words, if $Y_0(K) \simeq N\#S^2 \times S^1$ and $Y \in \mathbb{Q}HS^3$, then K is the unknot. The converse is also true: by definition, the unknot by definition must bound a disk, and so is nullhomotopic in Y. In particular, this disk allows us to zoom down into a B^3 neighborhood of K in Y to perform 0-surgery. This creates an $S^2 \times S^1$ summand which is connect sum to the complement of the B^3 neighborhood.

Remark 4. Nullhomologous means that K is the boundary of some two cell, nullhomotopic means that it can be continuously deformed through the space into a point. All nullhomotopic knots are nullhomologous, but not all nullhomologous knots are nullhomotopic. In S^3 these are equivalent, but in other manifolds this may not be the case. In fact, all knots in S^3 are nullhomotopic, because $\pi_1(S^3) = H_1(S^3) = 0$. But in other manifolds, just because a knot is nullhomotopic, does not mean that this gives an isotopy to the unknot. Being the unknot means it bounds a disk, which is an even more particular condition on nullhomotopy, since the contraction map should cut out a disk, not just any old weird space (e.g. think of a shrinking trefoil).

2 Gabai's Property R

Theorem 2.1 (Gabai, '87)**.** $S^3_0(K) \simeq S^2 \times S^1$ if and only if K is the unknot.

Proof.

1. If S is a minimal genus Seifert surface for a knot k in S^3, then there exists a taut finite depth foliation F of $S^3 \setminus N(K)$ such that S is a leaf of F and $F|_{\partial N(K)}$ is a foliation by circles.
2. Doing 0-surgery along K is equivalent to filling each longitude of the knot complement in with disks. Relative to the leaves, what this means is that we are attaching to each leaf a disk, closing up the submanifolds.
3. The manifold $Y_0(K)$ obtained by performing 0-surgery to a knot $K \hookrightarrow S^3$ has a taut finite depth foliation F such that K (viewed in M) is transverse to F and intersects every leaf of F. In particular, F has a compact leaf S of genus equal to the genus of K.
4. $M := Y_0(K)$ is obtained by performing 0-surgery on a knot K in S^3, then M is prime and
$$g(K) = \min\{g(S) \mid S \text{ is an embedded, oriented, nonseparating surface}\}$$
5. It is this last fact that causes K to be the unknot. $S^2 \times S^1$ is prime, and we of course see that there is an S^2 nonseparating surface in it. So $g(K) \leq 0 \implies g(K) = 0$. So K is the unknot.

□

3 Hom and Lidman's Generalization

Hom and Lidman's proof makes use of Heegaard Floer Homology with twisted coefficients. Before we get into what that means, let me just state the result that they proved:

Theorem 3.1. Let $Y \in \mathbb{Q}HS^3$ and K a nullhomologous knot in Y. Suppose that $Y_0(K) = N\#S^2 \times S^1$. Then if $\dim(\widehat{HF}(N)) = \dim(\widehat{HF}(Y))$, then $N = Y$ and K is the unknot. Otherwise, $\dim(\widehat{HF}(N)) \leq \dim(\widehat{HF}(Y))$.

Before talking about the proof, let me say why this implies theorem 1.3. Suppose that K is a nullhomotopic knot in Y such that $Y_0(K)$ contains a nonseparating 2-sphere. Then in particular, by remark 3, we know that $Y_0(K)$ must have a $S^2 \times S^1$ summand. In ADD SOURCE, under these hypotheses it's shown that N must be Y, i.e. $Y_0(K) \simeq Y\#S^2 \times S^1$. So we know that we are in the case that $\dim(\widehat{HF}(N)) = \dim(\widehat{HF}(N))$. Therefore by theorem 1.3, K must be the unknot.

3.1 Knot Floer Homology

Definition 3.2 (Knot Handlebody)**.** The standard diagram for a knot projection. Fix a knot projection D for K in R^2 , together with a distinguished edge adjoining the infinite region in the projection complement. The edge is distinguished by placing a star somewhere on the edge. We call this data a decorated knot projection of K. To a decorated knot projection, we can associate a Heegaard diagram representing K, as follows. First, singularize the projection, so that the crossings are actually double-points. Next, take a regular neighborhood of the resulting planar graph, to obtain a handlebody H embedded in $\mathbb{R}^2 \subset S^3$. The regions in the complement of the graph in the plane have two distinguished regions that adjoin the marked edge, one of which is the infinite region in R^2 . For each bounded region in the graph complement, there is a corresponding α-circle. In a neighborhood of each crossing, we associate a β-circle.

Definition 3.3. Let $K \subset S^3$ be an oriented knot. There are several different variants of the knot Floer homology of K. The simplest is the hat version, which takes the form of a bi-graded, finitely generated Abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,s\in\mathbb{Z}} \widehat{HFK}_i(K)$$

Suppose $K \subset Y$ is a null-homologous knot in a rational homology 3–sphere, F is a fixed Seifert surface. There is a compatible doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for the knot K. This gives rise to a map from intersection points between the two tori $T_\alpha \cap T_\beta$ to relative Spinc structures on $Y \setminus K$ $s_{w,z} : T_\alpha \cap T_\beta \to Spinc(Y_0(K))$. For each Spinc structure s on Y , the knot chain complex $C(s) = CFK^\infty(Y, K; S)$ is a free abelian group generated by $[x, i, j] \in (T_\alpha capT_\beta) \times \mathbb{Z}^2$, where the ∂ map is a certain count of holomorphic whitney disks taking into account the bigrading.

3.2 HFH Mapping Cone Formula

3.3 Proof of theorem

General Idea. 1. Key fact:

$$\boxed{g(K) = \min\{s \mid \widehat{v}_{i,\star} \text{ is an isomorphsim for all } i \geq s, t \in \mathrm{Spin}^c(Y)\}}$$

2. Idea is to show that if $Y_0(K)$ has an $S^2 \times S^1$ summand, then a certain homologically defined map $\widehat{v}_{0,\star} + \widehat{h}_{s,\star}$ has to be 0. Basically, your maps $\widehat{v}$ and $\widehat{h}$ are defined in such a way that

$$HF^\circ(Y_0(K), t_s) \cong H_*(\mathrm{Cone}(v_s^\circ + h_s^\circ))$$

3. This map is a sum of two maps, and since we're in $\mathbb{F}_2$, it must be then that these two maps are equal: $\widehat{v}_{0,\star} = \widehat{h}_{s,\star}$

4. Raising this to twisted coefficients, one can show that the map $\widehat{v}_{0,\star} + T\widehat{h}_{s,\star}$ is an isomorphism. But since $\widehat{v}_{0,\star} = \widehat{h}_{s,\star}$, $\widehat{v}_{0,\star} + T\widehat{h}_{s,\star}$ is an isomorphism, so $\widehat{v}_{0,\star}$ is an ismorphism.

5. So if you can show that $\widehat{v}_{0,\star}$ is an isomorphism, then by the factoring property of these maps $\widehat{v}_s$ on a chain level, then $\widehat{v}_{s,\star}$ must be surjective.

6. Then if you can show that $\dim(\mathrm{dom}(\widehat{v}_{s,\star})) = \dim(\mathrm{codom}(\widehat{v}_{s,\star}))$, surjectivity plus this tells you that $\widehat{v}_{s,\star}$ is also an isomorphism.

7. Showing that the dimensions are the same is also a homological argument, combining the idea of the homology of the cone with the other key fact, which is that $HF^+(Y_0(K), t_s; \mathbb{F}[[T, T^{-1}) = 0$ - the HF^+ homology of a 0-surgery vanishes over twisted coefficients:

> If M is a three-manifold which contains a non-separating two-sphere S, then
>
> $$HF^\circ(M; \mathbb{F}[[T, T^{-1}]) = 0,$$
>
> where T denotes a generator of H_1 of the $S^2 \times S^1$ summand.

4 Li's Generalization