Morse Homology

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1 Definition of a Morse Function

Let M be a smooth, (closed) n-dimensional manifold, and suppose that we have some smooth function $f : M \to \mathbb{R}$. A point $x \in M$ is said to be a **critical point** of f if $df_x = 0$. We denote the set of critical points of a function by $\mathrm{Crit}(f)$. Equip M with some random connection ∇. Given some $x \in \mathrm{Crit}(f)$, we can define the **Hessian** of f at x to be the map $\mathrm{Hess}(f)_x : T_xM \mapsto T_x^*M$, given by

$$\mathrm{Hess}_x(f)(v) := \nabla_v(df).$$

Notice that since df vanishes at x, and ∇ behaves covariantly, the choice of ∇ is irrelevant. Denoting local coordinates on a chart of M by $\{x^i\}$, one can show that the Hessian is an $n \times n$ symmetric matrix with entries $\partial^2 f/\partial_i\partial_j$.
The Hessian is a linear map $T_xM \to T_x^*M$, but equipping M with a metric allows us to identify $T_xM \simeq T_x^*M$, so that Hess defines a symmetric bilinear form. A critical point x of f is **nondegenerate** if $\mathrm{Hess}(x)$ is invertible. The **Morse index** of a critical point x, denoted $\mathrm{ind}(x)$, is the number of negative eigenvalues of $\mathrm{Hess}(x)$.
With this comes our first definition:

Definition 1.1 (Morse Function)**.** Let $f : M \to \mathbb{R}$ be a smooth function. Then f is said to be Morse if all of its critical points are nondegenerate.

1.1 Ascending and Descending Manifolds of a Morse Function

Let $f : M \to \mathbb{R}$ be a Morse function. We want to use f to study the underlying topology of the manifold M. Equip M with an arbitrary metric. Then we can define a "gradient like" vector field V on M which locally looks like the gradient of f with respect to g. We define the vector field locally as

$$V = -\mathrm{grad}(f) = -\nabla^g f = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n}\right),$$

Or globally as the vector field V satisfying

$$g(V, Y) = -df(Y).$$

By construction, the gradient vector field vanishes only when $df = 0$, i.e. at critical points, and f decreases along its integral curves. We can combine these two properties and study them in the following way:

Definition 1.2. For the vector field $-\mathrm{grad}(f)$, there are two submanifolds associated to f arising from its flow lines ψ^t. More precisely, let $\psi : M \times [0, 1] \to M$ denote the flow of $-\mathrm{grad}(f)$. For a critical point $p \in \mathrm{Crit}(f)$, the **descending manifold** of p is the set

$$\mathcal{D}(p) := \left\{ y \in M \;\middle|\; \lim_{s \to -\infty} \psi_s(y) = p \right\}$$

and the **ascending manifold** of p is the set

$$\mathcal{A}(p) := \left\{ y \in M \;\middle|\; \lim_{s \to +\infty} \psi_s(y) = p \right\}.$$

Indeed, each of these are in fact submanifolds, and the dimension of each is given by

$$\dim(\mathcal{D}(p)) = \operatorname{ind}(p) = \operatorname{codim}(\mathcal{A}(p)).$$

To see this, notice that by construction $\mathcal{D}(p)$ and $\mathcal{A}(p)$ deformation retract onto p, and so have the homotopy type of a point. They must be homeomorphic to an open neighbourhood of the point in M. They are defined in a smooth way though, so $\mathcal{D}(p)$ and $\mathcal{A}(p)$ are actually immersed open disks. To deduce their dimension, it is enough to look at the dimension of the image of the tangent space of the immersed submanifold. Looking at a local chart about p, the dimension of $\mathcal{D}(p)$ is given by the number of negative eigenvalues of the Hessian at p. The dimension of $\mathcal{A}(p)$ is given by the number of positive eigenvalues of the Hessian at p.
Notice that integral curves of the gradient vector field can only begin and end in critical points in the limit. A **flow line** from p to q is a path $\gamma : \mathbb{R} \to M$ with $\gamma'(s) = V(\gamma(s))$, and $\lim_{s\to-\infty} \gamma(s) = p$ and $\lim_{s\to\infty} \gamma(s) = q$. That is, γ is an infinite path in M, $\gamma : \mathbb{R} \to M$ connecting critical points p and q, contained in the intersection $\mathcal{D}(p) \cap \mathcal{A}(p)$ and flowing along V. There is a natural action of $\mathbb{R}$ on a flow line by precomposition by translation of $\mathbb{R}$. We denote the set of flow lines modulo translation between p and q by

$$\mathcal{M}(p,q) = {}^{\mathcal{D}(p)\cap\mathcal{A}(q)}\!\big/_{\mathbb{R}}.$$

Understanding the relationship between critical points can be explored through the flow lines connecting them. However, to make sense of $\mathcal{M}(p,q)$, we need to impose a condition. We say that the pair (f,g) is **Morse-Smale** if f is a Morse function, and for any $p,q \in \operatorname{Crit}(f)$, $\mathcal{D}(p)$ and $\mathcal{A}(q)$ intersect transversely.

Example. Morse functions on the torus, or any other nice manifold, are generally studied by embedding the manifold into $\mathbb{R}^n$ for some n and then projecting onto one of the coordinates. The most obvious choice of a height function on a torus is, alas, not Morse-Smale: the ascending and descending manifolds of p and q do not intersect transversely.

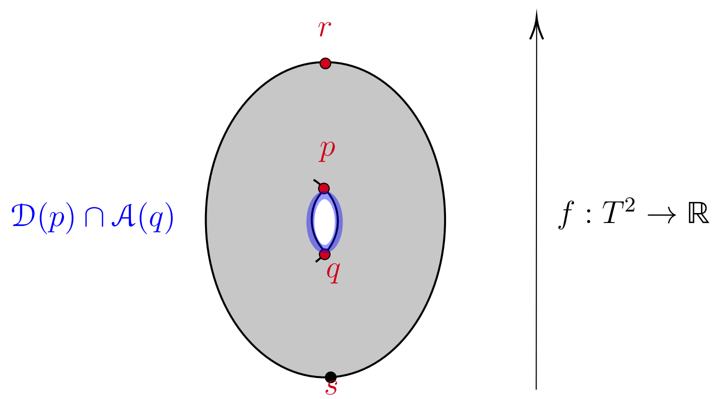

Figure 1: A Morse function on T^2 that is not Morse-smale.

However, if we tilt the torus slightly and then take the height function of the resulting embedding into $\mathbb{R}^3$, this now becomes a Morse-Smale pair. One can show that this condition holds generically though, so no need to worry.

If (f,g) is a Morse-Smale pair, then for all pairs $p,q \in \operatorname{Crit}(f)$, $\mathcal{D}(p)$ and $\mathcal{A}(q)$ intersect etransversely, so that $\mathcal{D}(p) \cap \mathcal{A}(q)$ is a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. Quotienting by this $\mathbb{R}$

action, it follows that for a Morse-Smale pair, $\mathcal{M}(p,q)$ is a smooth manifold of dimension

$$\dim(\mathcal{M}(p,q)) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1.$$

One can equip $\mathcal{M}(p,q)$ with an orientation that is compatible with that of $\mathcal{D}(p)$.

1.2 Compactification of the Moduli Space of Flow Lines

Suppose we have (f,g) a Morse-smale pair. Let's think about a very specific case of a pair of critical points: when $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$. Then $\dim(\mathcal{M}(p,q)) = 1 - 1 = 0$. This is a nice situation to be in, since to understand the flow lines between p and q, the only data we really need is the count of these points. However, counting the number of points of an arbitrary 0-dimensional manifold is not always clear - what we need to do is compactify the space in some natural way, so that we can count something finite.

Theorem 1.3. Let M be a closed manifold and (f,g) a Morse-Smale pair. Then for any two critical points p,q, the moduli space of flow lines $\mathcal{M}(p,q)$ has a natural compactification to a manifold with smooth corners $\overline{\mathcal{M}}(p,q)$, with codimension-$k$ corners

$$\overline{\mathcal{M}}(p,q)_k = \bigcup_{\substack{r_1,\ldots,r_k \in \mathrm{Crit}_i(f) \\ p,r_1,\ldots,r_k,q \text{ distinct}}} \mathcal{M}(p,r_1) \cdots \times \ldots \mathcal{M}(r_k,q).$$

The proof of this result relies on understanding sequences of flow lines in M. We can compactify $\mathcal{M}(p,q)$ by sequentially compactifying it: given a sequence of flow lines, we need to add in all possible things that it (or a subsequence of it) may converge to.
The main part of the result can be boiled down to the following:

Proposition 1.4. Let $\{\gamma_n\}$ be a sequence of flow lines from p to q, and let $\hat{\gamma} = (\hat{\gamma}_0, ..., \hat{\gamma}_k)$ be a k-times broken flow line from $p \to q$. Define $\lim_{n\to\infty}[\gamma_n] = [\hat{\gamma}]$ to mean that for each n, there exist real numbers $s_{n,0} < s_{n,1} < ... < s_{n,k}$ such that $\gamma_n(s_{n,i} + \_) \to \hat{\gamma}_i$ in $\mathbb{C}^\infty$ on compact sets. Then any sequence of flow lines $\{\gamma_n\}$ from p to a has a subsequence which converges to some k-times broken flow line as above for some $k \geq 0$.

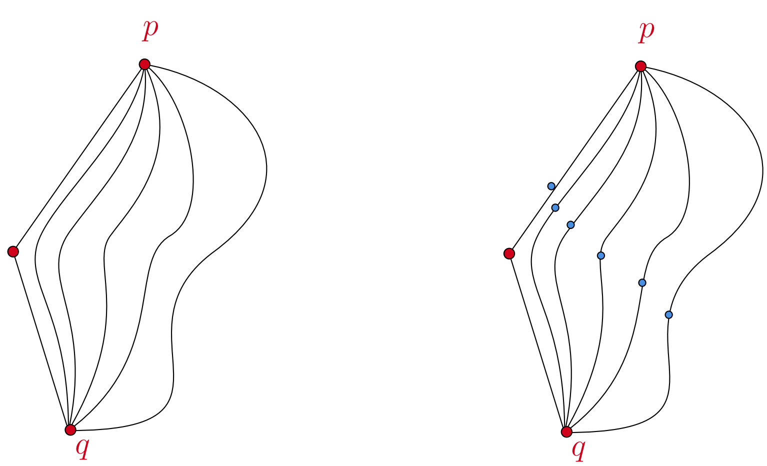

Figure 2: Sequence of flow lines converging to a broken flow line.

Before we get to the proof, the point of this proposition is that the addition of all broken flow lines between p and q to $\mathcal{M}(p,q)$ makes the space sequentially compact, which is equivalent to compactness. Moreover, this compactification sits nicely inside our manifold M, and so is easy to study.

Proof. This is more-so a sketch of the argument. Let $\{\gamma_n\}$ be a sequence of flow lines from p to q. The key idea is that sequential compactness is a local property that can only break down around critical points. Moreover, away from critical points, flow lines can only converge to flow lines, since really they're complete integral curves of a vector field, and so any limit of a sequence of flow lines must be a flow line by standard ODE theory. Hence proceed as follows. For all critical points of intermediate index between p and q, identify those have some small neighbourhood that does not intersect the sequence. Throw those out, and only look at the intermediate critical points who do not have such a neighbourhood. Order the finitely many remaining intermediate critical points $r_1, ..., r_k$ from p to q by index, and arbitrary ordering for those of the same index. Choose local coordinates around r_1 that gives a Morse chart. The sequence of flow lines restricted to this chart must have a convergent subsequence that converges to some path through r_1. This follows from the sequential compactness of $\mathbb{R}^n$, the fact that the Morse chart is centered on r_1, and the fact that it's enough to check sequential compactness pointwise. Away from r_1, the path is the limit of a sequence of restricted flow lines, hence by uniqueness properties of solutions to ODEs it must itself be a restricted flow line (an integral curve). With this new subsequence, iterate this process for r_2, then r_3 and so on, until one obtains a subsequence that converges on Morse charts around intermediate critical points to fragments of a "broken flow line". The real numbers from the definition of converging flow lines correspond to pushing each flow line γ_n along (reparametrizing) so that it "jumps" above the zero in the limit.

□

Hence, $\mathcal{M}(p,q)$ has a natural compactification $\overline{\mathcal{M}}(p,q)$, which is 0-dimensional manifold itself, and hence is a finite collection of points. But $\mathcal{M}(p,q) \subseteq \overline{\mathcal{M}}(p,q)$ as sets, so this forces $\mathcal{M}(p,q)$ to be finite. We may now use this to relate critical points to each other.

2 The Morse Complex

We can combine the previous work to define a topological invariant of a manifold, namely Morse homology. The Morse complex (C^M, ∂^M) is defined as follows. The chain module C_i^M is the free abelian group generated by the critical points of f of index i,

$$C_i^M(M) := \mathbb{Z}\langle \mathrm{Crit}_i(f) \rangle.$$

The boundary map $\partial_i^M : C_i^M \to C_{i-1}^M$ is defined as

$$\partial_i^M(p) = \sum_{q \in \mathrm{Crit}_{i-1}(f)} \#\mathcal{M}(p,q) \cdot q,$$

where $\#\mathcal{M}(p,q)$ is the count of the number of flow lines from p to q.
The first order of business is making sure that this is indeed a chain complex, i.e. that $(\partial^M)^2 = 0$.

Proposition 2.1 (Well-definedness)**.** (C_i^M, ∂_i^M) is a well-defined chain.

Proof. Let $p \in C_i^M(M)$ be a critical point. Then we'd like to show that

$$\partial_{i-1}^M \partial_i^M(p) = 0.$$

By definition, we have that

$$\begin{aligned}
\partial_{i-1}^M \partial_i^M(p) &= \partial_{i-1}^M \left(\sum_{r \in \mathrm{Crit}_{i-1}(f)} \#\mathcal{M}(p,q) \cdot r \right) \\
&= \sum_{r \in \mathrm{Crit}_{i-1}(f)} \#\mathcal{M}(p,r) \cdot \partial_{i-1}^M(r) \\
&= \sum_{r \in \mathrm{Crit}_{i-1}(f)} \sum_{q \in \mathrm{Crit}_{i-2}(f)} \#\mathcal{M}(p,r)\#\mathcal{M}(r,q) \cdot q \\
&= \sum_{q \in \mathrm{Crit}_{i-2}(f)} \#\partial(\overline{\mathcal{M}}(p,q)) \cdot q \\
&= \sum_{q \in \mathrm{Crit}_{i-2}(f)} 0 \cdot q \\
&= 0,
\end{aligned}$$

using the fact that the signed count of the number of points in a compact manifold is 0. □
The catchphrase here is that "pairs of connecting orbits come in pairs".

Definition 2.2 (Morse Homology)**.** Let M be a manifold, and (f, g) a Morse-Smale pair. Then the **Morse homology** of M (wrt. f) is the homology of the chain complex (C_i^M, ∂_i^M),

$$H_i^M := {}^{\ker(\partial_i)}\!\big/\!_{\mathrm{Im}(\partial_{i-1})}.$$

3 Invariance of Morse Homology

3.1 Chain Homotopy

Let (f_0, g_0) and (f_1, g_1) be two Morse-Smale pairs for a manifold M. We will see in a later section that the Morse homology groups arising from these pairs are isomorphic to the ones from singular homology. This is an indirect way to prove that Morse homology is invariant under choice of Morse function. Morse homology should, however, admit an explicit proof of this result. This chapter is devoted to doing exactly that.
The idea is to take a more geometric approach, extending the ideas that we've already encountered along the way to relate the two chain complexes together. The idea will be to "flow" along a path between (f_0, g_0) and (f_1, g_1), and keep track of what it does to the critical points and the flow lines. More precisely, let

$$\Gamma := \{(f_t, g_t) \ |t \in [0,1]\}$$

be a one-parameter family of smooth functions and metrics from (f_0, g_0) to (f_1, g_1). Denote the chain complexes associated to (f_0, g_0) and (f_1, g_1) by (C_i^0, ∂_i^0) and (C_i^1, ∂_i^1) respectively. The notes that I was reading in conjunction to this described everything in the language of category theory, which seemed to make a bit more sense conceptually. You can think about a category of Morse-Smale pairs for a fixed manifold M, with morphisms given by homotopy equivalence classes of paths between the pairs. The goal is to define a functor Φ from this category to the category of Morse chain complexes on M. The map between objects is simple: just take a Morse-Smale pair and map it to the Morse chain complex that it gives rise to. The interesting part comes when we think about the morphisms.

Remark. Before we get into the detail, it's worth discussing what is going on heuristically. If we can find a path Γ from (f_0, g_0) to (f_1, g_1), then *homotopically* there's an inverse path going the other direction, $-\Gamma$. If we can use Γ to construct a map Φ^Γ between chain complexes C^0 and C^1, which depends only on Γ's homotopy class and behaves well under concatenation, then we will have an isomorphism Φ^Γ on homology.

Let us return to our path $\Gamma = \{(f_t, g_t) \mid t \in [0,1]\}$ between (f_0, g_0) and (f_1, g_1). The idea is to beef the f_t up into something that looks like a Morse function on $M \times [0,1]$ and then study the flow lines on this manifold. One way we could try to do this is use the t coordinate to interpolate between the different functions in some smooth way. With the general sign conventions we've chosen however, it makes sense to cut to the chase and simply define a vector field on $[0,1] \times M$:

$$V := (1-t)(t)(1+t)\partial_t + V_t,$$

where V_t is the negative gradient of $f_t : M \to \mathbb{R}$. The critical points of V correspond to the critical points of V_t when $t = 0$ and $t = 1$. The concepts of descending/ascending manifolds, moduli spaces of flow lines and transversality all extend to this case, which allow us to associate to each Γ a chain map sending $p \in \mathrm{Crit}_i(f_0)$ to

$$\Phi_\Gamma(p) = \sum_{q \in \mathrm{Crit}_i(f_1)} \#\mathcal{M}((0,p),(1,q)) \cdot q.$$

Here, $\mathcal{M}((0,p),(1,q))$ is the moduli space of flow lines (mod parametrization) going from the critical point of index $i+1$ $(0,p)$ to the critical point of index i $(1,q)$ in $[0,1] \times X$. Notice that by construction, if $(0,p)$ has index $i+1$ then p must have index i, and if $(1,q)$ has index i then q must have index i.

Proposition 3.1. The map $\Phi_\Gamma : C_i^0 \to C_i^1$ is a chain map.

Proof. This is really just an exercise in analysis, with a touch of geometric insights thrown in. We need to show that $\Phi_\Gamma \circ \partial_i^0 = \partial_i^1 \circ \Phi_\Gamma$. Let $p \in \mathrm{Crit}_i(f_0)$. By definition,

$$\partial_i^0(p) = \sum_{q \in \mathrm{Crit}_{i-1}(f_0)} \#\mathcal{M}(p,q) \cdot q,$$

Hence

$$\begin{aligned}
\Phi_\Gamma(\partial_i^0(p)) &= \sum_{r \in \mathrm{Crit}_{i-1}(f_0)} \#\mathcal{M}(p,r) \cdot \Phi_\Gamma(r) \\
&= \sum_{r \in \mathrm{Crit}_{i-1}(f_0)} \#\mathcal{M}(p,r) \sum_{q \in \mathrm{Crit}_{i-1}(f_1)} \#\mathcal{M}((0,r),(1,q)) \cdot q \\
&= \sum_{q \in \mathrm{Crit}_{i-1}(f_1)} \left[\sum_{r \in \mathrm{Crit}_{i-1}(f_0)} \#\mathcal{M}(p,r) \#\mathcal{M}((0,r),(1,q)) \right] \cdot q \qquad (\dagger)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\partial_i^1(\Phi_\Gamma(p)) &= \sum_{r \in \mathrm{Crit}_i(f_1)} \#\mathcal{M}((0,p),(1,r)) \sum_{q \in \mathrm{Crit}_i(f_1)} \#\mathcal{M}(r,q) \cdot q \\
&= \sum_{q \in \mathrm{Crit}_{i-1}(f_1)} \left[\sum_{r \in \mathrm{Crit}_i(f_1)} \#\mathcal{M}(r,q) \#\mathcal{M}((0,p),(1,r)) \right] \cdot q \qquad (\ddagger)
\end{aligned}$$

To prove the statement, we thus need to show that the coefficients of q in $\dagger$ and $\ddagger$ are equal. Let's group interesting terms together:

$$\begin{aligned}
\Phi_\Gamma(\partial_i^0(p)) &= \sum_{q \in \mathrm{Crit}_{i-1}(f_1)} \left[\sum_{r \in \mathrm{Crit}_{i-1}(f_0)} (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)} \#\mathcal{M}((0,p),(0,r)) \#\mathcal{M}((0,r),(1,q)) \right] \cdot q \\
&= \sum_{q \in \mathrm{Crit}_{i-1}(f_1)} \left[\sum_{r \in \mathrm{Crit}_{i-1}(f_0)} (-1) \#\mathcal{M}((0,p),(0,r)) \#\mathcal{M}((0,r),(1,q)) \right] \cdot q. \qquad (\dagger)
\end{aligned}$$

On the other hand,

$$\partial_i^1(\Phi_\Gamma(p)) = \sum_{q \in \mathrm{Crit}_{i-1}(f_1)} \left[\sum_{r \in \mathrm{Crit}_i(f_1)} \#\mathcal{M}((1,r),(1,q) \#\mathcal{M}((0,p),(1,r)) \right] \cdot q \qquad (\ddagger)$$

Let A be the coefficient of q from $\dagger$ and B be the coefficient of q from $\ddagger$. Note that

$$B - A = \#\partial\overline{\mathcal{M}((0,p),(1,q))}.$$

But the signed count of the points in the boundary of a compact manifold vanishes, Hence $B - A = 0$, so $A = B$. The proposition follows.

□

We've shown that a path between two Morse-Smale pairs induces a chain map between their chain complexes, but we still have some work to do. We need to justify that this is in fact an *isomorphism* on the level of homology. This follows immediately from the next two propositions, which tells us that two homotopic paths Γ_1 and Γ_2 induce chain maps Φ_{Γ_1} and Φ_{Γ_2} that are chain homotopic, and $\Phi_{\Gamma_2 * \Gamma_1}$ is chain homotopic to $\Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}$.

3.2 Properties of Chain Homotopy

We'll begin with the first result.

Proposition 3.2. Let Γ_1 and Γ_2 be two paths between (f_0, g_0) and (f_1, g_1) that are endpoint-fixed homotopic in the space of paths on Morse-Smale pairs. Then Φ_Γ depends only on the homotopy type of Γ, i.e. that there exists a chain homotopy between Φ_{Γ_1} and Φ_{Γ_2}.

Proof. We can think of a homotopy between Γ_1 and Γ_2 as a family of functions and metrics

$$\{(f_d, g_d) \mid d \in D\},$$

Where D is a digon. Recall that a digon is a bounded subset of $\mathbb{R}^2$ whose boundary comprises a simple, connected graph with two vertices and two edges. The picture to have in mind is given below.

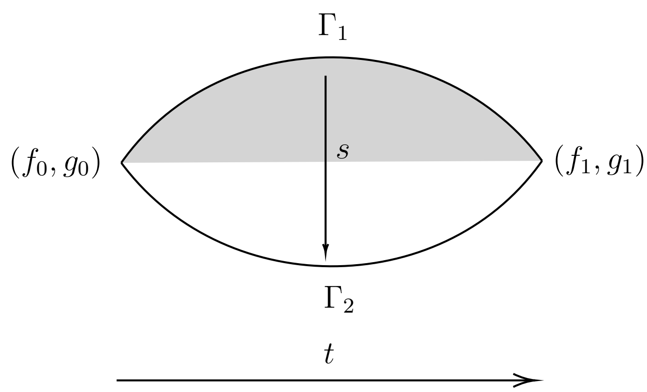

Figure 3: Digon is a 1-parameter family of paths between (f_0, g_0) and (f_1, g_1).

The argument for why this in fact gives rise to a chain homotopy $\Phi_{\Gamma_1} \to \Phi_{\Gamma_2}$ has a very similar flavour to the previous proposition. Let $\hat{g}$ be a metric on D such that the edges have length 1, and let $\hat{f} : D \to \mathbb{R}$ be a function with an index 2 critical point on the left vertex, and an index 0 critical point on the other. We also require the negative gradient of $\hat{f}$ with respect to $\hat{g}$ to be tangent to the edges and agree with the negative gradient of $\frac{1}{4}(t+1)^2(t-1)^2$. Denote the negative gradient of $\hat{f}$ with respect to $\hat{g}$ by $\hat{V}$. Then similarly to before, define a vector field on $D \times M$ by

$$V = \hat{V} + V_d,$$

where V_d is the negative gradient vector field of f_d with respect to g_d. Counting flow lines from (v_1, p) to (v_2, q) gives us a map $K : C_i^0 \to C_{i+1}^1$, and we can run similar analytical arguments, along with geometric insight about the boundary, to show that these maps do in fact form a chain homotopy. $\square$

3.3 Examples/Exercises

The following are either examples of calculating morse homology, chain maps, or exercises from Hutchings or Salamon.

Exercise. *Claim:* If $\Gamma = \{(f_t, g_t)\}$ is a constant family with (f_t, g_t) Morse-Smale, then Γ is admissible and $\Phi_\Gamma = \text{id}$.

Proof. Let V be a vector field on $[0,1] \times M$ given by

$$V = (1-t)t(1+t)\partial_t - \text{grad}(f),$$

where $-\text{grad}(f)$ is the negative gradient vector field of f with respect to g. Then the critical points of V of index i are $(0,p)$ and $(1,q)$, where $\text{ind}(p) = i-1$ and $\text{ind}(q) = i$. Schematically, the flow lines on $[0,1] \times M$ look like for the constant path. Given a critical point $(0,p)$ of V, the

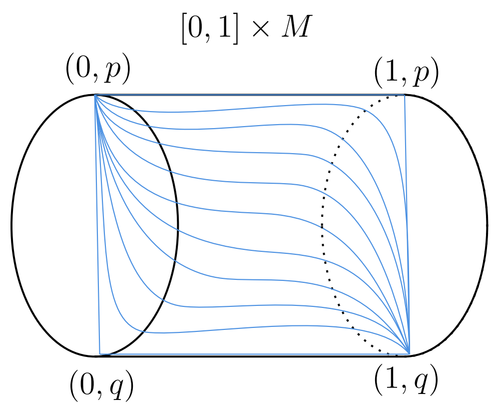

Figure 4: Flow lines on $[0,1] \times M$ for a constant path Γ.

associated descending manifold is a half open-disk homeomorphic to

$$\mathcal{D}((0,p)) \simeq \mathcal{D}(p) \times [0,1)$$

and similarly if $(1,q)$ is a critical point of V, the associated ascending manifold is an open half disk

$$\mathcal{A}((1,q)) \simeq (0,1] \times \mathcal{A}(q).$$

As a sanity check, notice that if $(0,p)$ is a critical point of index i, we'd expect $\mathcal{D}((0,p))$ took look like (half) of an open disk of dimension i. We know that $ind(p) = i-1$ in M, and so under this identification this dimension appears. One can similarly check that if $(1,q)$ is a critical point of index i in $[0,1] \times M$, then $\text{ind}(q) = i$ in M, so it has codimension i. We expect $(1,q)$ to have codimension $i-1$, which aligns with this calculation.
The first thing we need to do to prove this claim is show that Γ is admissible, i.e. the descending/ascending manifolds of V in $[0,1] \times M$ intersect transversely. We have two possibilities:

Case I) $\text{ind}((0,p)) \leq \text{ind}((1,q))$, in which case

$$\mathcal{D}((0,p)) \cap \mathcal{A}((1,q)) = \emptyset,$$

in which case the intersection is trivially transversal, or

Case II) $\mathrm{ind}((0,p)) > \mathrm{ind}((1,q))$, in which case

$$\mathcal{D}((0,p)) \cap \mathcal{A}((1,q)) \simeq (\mathcal{D}(p) \cap \mathcal{A}(q)) \times (0,1),$$

and the transversality of $\mathcal{D}((0,p)) \cap \mathcal{A}((1,q))$ follows from the assumed transversality of $\mathcal{D}(p) \cap \mathcal{A}(q)$.

Hence, Γ is admissible. For the second part of the claim, this follows immediately from considering the definition of Φ_Γ. For a critical point p of index i,

$$\Phi_\Gamma((0,p)) = \sum_{q \in \mathrm{Crit}_i(f)} \#\mathcal{M}((0,p),(1,q)) \cdot q,$$

Which by choice of path Γ we must have $\#\mathcal{M}((0,p),(1,q)) = \delta_q^p$. That is, $\Phi_\Gamma(p) = p$. $\square$

4 Morse Homology is Isomorphic to Singular Homology

5 Genericness and Transversality