Legendrian Surgeries

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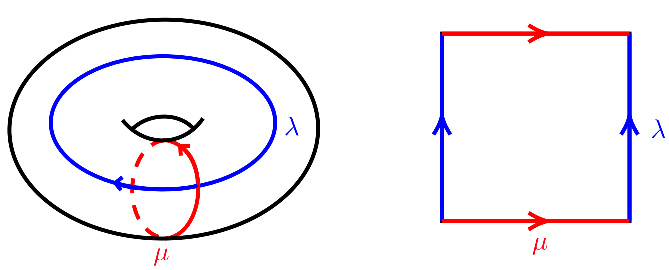
Contents

1 Topological Surgery

1.1 Dehn surgery

Recall the general procedure of performing Dehn surgery to a topological 3 manifold Y: Choose a knot $K \subseteq Y$, and let $\nu(K)$ denote its tubular neighborhood. Then, choose a meridian and longitude of the knot, considered as curves on $\partial\nu(K)$. The meridian μ should bound a disk in $\nu(K)$, and the longitude λ should be homotopically nontrivial in $\nu(K)$, or equivalently nullhomologous in the $Y \setminus \nu(K)$. Such a choice of μ and λ allows us to identify $\partial\nu(K)$ with $\mathbb{R}^2/\mathbb{Z}^2$, with where μ has slope 0 and λ has slope ∞. To perform Dehn surgery, cut out the tubular neighborhood, and glue it in back by an orientation preserving diffeomorphism of the boundary $S^1 \times S^1$.



By the above, the space of orientation-preserving diffeomorphisms can be identified with $SL_+(2, \mathbb{Z})$. Let $p/q \in \mathbb{Q}$. We refer to the above procedure sending $\mu \mapsto p\mu + q\lambda$ as rational p/q surgery.

1.2 Dehn surgery as handle attachments

We can also describe integral surgeries as handle attachments to a 4-manifold. Let $X = Y \times [0, 1]$. Then X is a 4-manifold with boundary $Y \sqcup -Y$. We may attach a 4-dimensional 2-handle $D^2 \times D^2$ to X along the boundary $Y \times \{1\}$ by specifying an attaching map $f : S^1 \times D^2 \hookrightarrow Y$. Such a map can be described by a knot in Y, along with a framing of the knot identifying its normal bundle with a tubular neighborhood of

the knot $\nu(K) \simeq S^1 \times D^2$, and a gluing map of the attaching region of the handle to $\nu(K)$. These gluing maps are indexed by the integers, $S^1 \times \{0\} \mapsto \lambda + k\mu$ and $\{0\} \times S^1 \mapsto \mu$. Here, μ denotes the meridian and λ the longitude chosen to identify $\nu(K)$ with $S^1 \times D^2$.
After attaching the 2-handle, the boundary $Y \times \{1\}$ loses a copy of $S^1 \times D^2$, and gains a copy of $D^2 \times S^1$. The cutting out of $S^1 \times D^2$ swaps the roles of μ and λ, and so under the identification, $\partial D^2 \times \{pt\} \mapsto k\mu + \lambda$. Thus, this operation performs k-surgery on $Y \times \{1\}$.

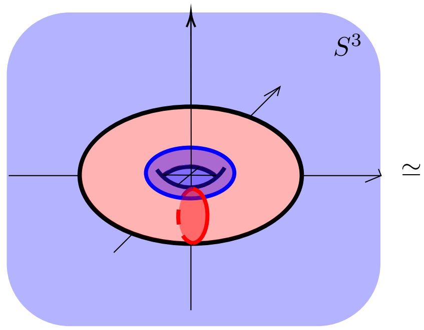
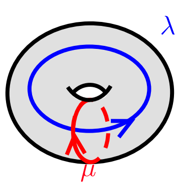
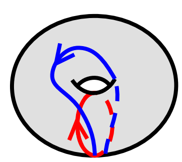
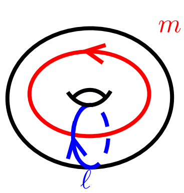
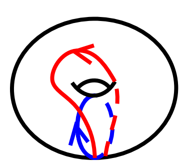


Figure 1: Meridian and longitude swap between knot neighborhood and knot complement.

1.3 Handle-attachings verus Dehn Fillings

When attaching a handle, we identify two **solid** tori with each other via an automorphism of $S^1 \times D^2$. When Dehn-filling, we **close** a boundary $S^1 \times S^1$ component by gluing in a copy of $S^1 \times D^2$. Such surgeries are given up to an automorphism of $S^1 \times S^1$. These processes are related by the following table.

Operation	Picture	Gluing Map	Matrix
Handle Attaching	 In manifold $\sim$  In handle	$\mu \mapsto \mu$, $\lambda \mapsto -\mu + \lambda$ Identifying two solid tori	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ related by transposition
Dehn Filling	 In knot complement $\sim$  In filling torus	$m \mapsto \mu - \lambda$, $\ell \mapsto \lambda$ Gluing along knot complement	$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

2 Contact surgery

We'd like to extend topological knot surgery to a contact/symplectic construction, equipping the surgered region with a contact structure that that in $\nu(K)$ under the chosen gluing. There are several models to do describe this extension. We begin by describing the standard model of a symplectic 2-handle attachment.

2.1 Symplectic 2-handle attachment

This description is used in [HT11]. An alternative description is given in [Gei08, Ch6], and [Etn06]. This construction is originally due to Eliashberg [Eli90], and independently Weinstein [Wei91].

Consider $\mathbb{R}^4$ with coordinates $\{q_1, q_2, p_1, p_2\}$, and symplectic form $\omega = \sum dp_i \wedge dq_i$. The vector field

$$V = \sum_i -p_i \partial_{p_i} + 2q_i \partial_{q_i}$$

is Liouville with respect to this symplectic form, and transverse to the hypersurface $Y = \{p_1^2 + p_2^2 = 1 \subseteq \mathbb{R}^4\}$.
Proof. Since ω is closed, $d\omega = 0$, so $\mathcal{L}_V \omega = d\iota_V \omega$. But

$$\begin{aligned} \iota_V \omega &= -p_1 dq_1 - p_2 dq_2 - 2q_1 dp_1 - 2q_2 dp_2, \\ \implies d(\iota_V \omega) &= -p_1 \wedge dq_1 - dp_2 \wedge dq_2 - 2dq_1 \wedge dp_1 - 2dq_2 \wedge dp_2 \\ &= dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = \omega. \end{aligned}$$

It is clearly also transverse to Y. $\square$
As such, $\lambda = \iota_V \omega$ is a contact for the hypersurface Y, given by

$$\lambda = -p_1 dq_1 - p_2 dq_2 - 2q_1 dp_1 - 2q_2 dp_2.$$

Within Y, we have a model Legendrian knot given by

$$K = \{q_1 = q_2 = 0, p_1^2 + p_2^2 = 1\}$$

Proof. We will switch to coordinates $p_1 = \cos(\theta)$ and $p_2 = \sin(\theta)$ for simplicity. Then our knot K admits the parametrization $\psi : (q_1, q_2, \theta) \mapsto (0, 0, \cos(\theta), \sin(\theta)) \in \mathbb{R}^4$, and our contact form with respect to these coordinates is

$$\lambda = -\cos(\theta) dq_1 - \sin(\theta) dq_2 + 2q_1 \sin(\theta) d\theta - 2q_2 \cos(\theta) d\theta.$$

In particular, when $q_1 = q_2 = 0$, the contact form becomes

$$\lambda = -\cos(\theta) dq_1 - \sin(\theta) dq_2,$$

for which certainly $\partial_\theta \in \ker(\lambda)$. So K is Legendrian. $\square$
We now describe a model of attaching a 2-handle to the region $\{p_1^2 + p_2^1 \geq 1, (q_1, q_2) \in \mathbb{R}^2\}$ as follows. The boundary of this region is a copy of $S^1 \times \mathbb{R}^2$, within which we specify an attaching region to be

$$\{p_1^2 + p_2^2 = 1, q_1^2 + q_2^2 \leq \epsilon\}$$

and consider our handle as

$$D^2 \times D^2 = \{p_1^2 + p_2^2 \leq 1, q_1^2 + q_2^2 \leq \epsilon\}.$$

Remark 1. Let $K : S^1 \hookrightarrow Y$ be a Legendrian knot in a cooriented contact manifold, with contact form λ. Such a contact form induces an orientation on the symplectic vector bundle $(\xi, d\lambda)$, and on the manifold Y, and these orientations are compatible. In a tubular neighborhood of K, there exist local coordinates such that one direction of the contact structure is given by flowing along the knot, and the other is transverse to the knot. Equipped with this induced orientation, there is a well-defined notion of a positive and negative Legendrian push-off of K, $\ell_\pm$. The positive Legendrian push-off gives a canonical framing from the contact structure. That is, we consider K as having $tb(K)$ framing, where $tb(K)$ is the Thurston-Bennequin number of K. Note that the standard Seifert framing differs from the contact framing by $tb(K)$.

We now glue on the 2-handle by framing $tb(K) - 1$. That is, we identify the longitude ℓ_+ with a longitude of linking number $tb(K) - 1$ in the handle. The handle attachment reduces the number of right-handed meridional twists of the contact structure in the tubular neighborhood of the knot by 1.

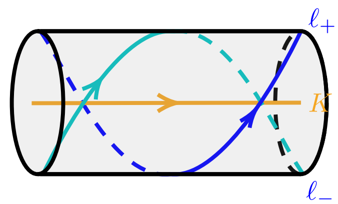


Figure 2: Positive and negative Legendrian push-offs

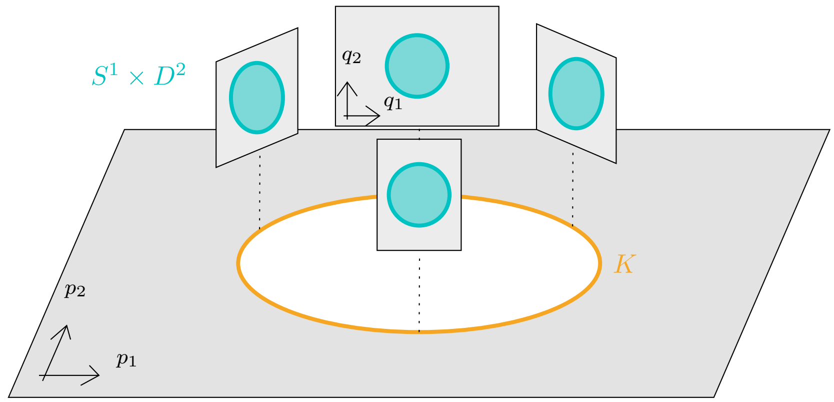


Figure 3: Model handle attaching region. Here we consider $\mathbb{R}^4$ as $\mathbb{R}^2_{q_1,q_2}$ fibered over $\mathbb{R}^2_{p_1,p_2}$.

We can round out the corners of the construction by making a suitable identification of the corner region with a smoothing hypersurface. On the thickened torus $\{p_1^2 + p_2^2 = 1, \epsilon/2 \leq q_1^2 + q_2^2 \leq \epsilon\}$, we can replace the hypersurface on the handle with one cut out by an equation given

$$f(q_1^2 + q_2^2, p_1^2 + p_2^2) = 0$$

where $\partial_x f > 0$ and $\partial_y f = 0$, so that the attaching looks like

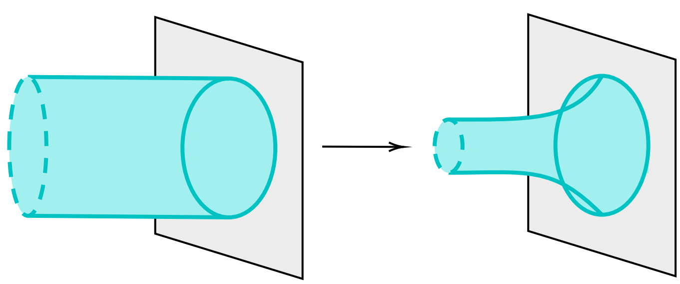


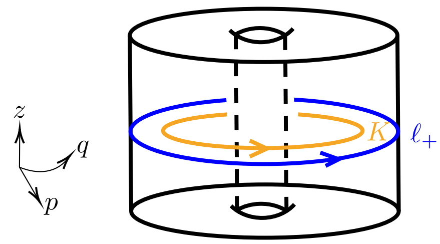
Figure 4: Smooth attaching of handle

Remark 2. The handle attaching amounts to doing -1-surgery on the knot K in Y. This description is useful, as it provides a canonical cobordism between Y before and after surgery. These cobordisms give rise to chain maps in SWH, ECH, and HFH.

2.2 Dehn surgery via 1-jet bundles

This description is used in [Avd23], and also appears in less detail in [BEE12]. The following exposition follows the notation of the former.

We may also describe Dehn surgery at the level of 3-manifolds. Whereas in the previous description we only obtained -1-surgery (and more generally integral surgeries), we can now extend these ideas to perform rational $1/k$-surgeries along Legendrian knots.
As before, let $K : S^1 \hookrightarrow Y$ be a Legendrian knot. We may consider an alternative description of a standard tubular neighborhood of K, identifying it with the 1-jet bundle of S^1, $\mathbb{R}_z \times T^*S^1(q,p)$, equipped with the contact form $\lambda = dz + pdq$. We restrict our attention to the neighborhood $\nu_\epsilon(K) := I_{\epsilon,z} \times I_{\epsilon,p} \times S^1_q$, where $I_\epsilon = [-\epsilon, \epsilon]$. We can identify K with the Legendrian $\{z = p = 0\}$. Moreover, the vector field $\partial_p \in \ker(\lambda)$ is transverse to this curve, and gives a canonical positive and negative push-off as before.



We'd like to construct agluing map that identifies the boundary of $\nu_\epsilon(K)$ with itself, sending the longitude $\ell_+ \mapsto \ell_+ - \mu$, where μ is a standard meridian. We will concentrate the nontriviality of the gluing at the top of the neighborhood. This will make the effect of doing surgery on the Reeb dynamics simpler to see.
Let $f : \mathbb{R} \to S^1$ be a smoothing of the piecewise linear function

$$f(x) = \begin{cases} 0 & \text{if } x < -1/2, \\ x + 1/2 & \text{if } -1/2 \leq x < 1/2, \\ 1 & \text{if } x \geq 1/2, \end{cases}$$

and let $\tau_f \in \mathrm{Diff}^+(\mathbb{R}_p \times S^1_q)$ be the diffeomorphism

$$\tau_f(p, q) = (p, q + f(p))$$

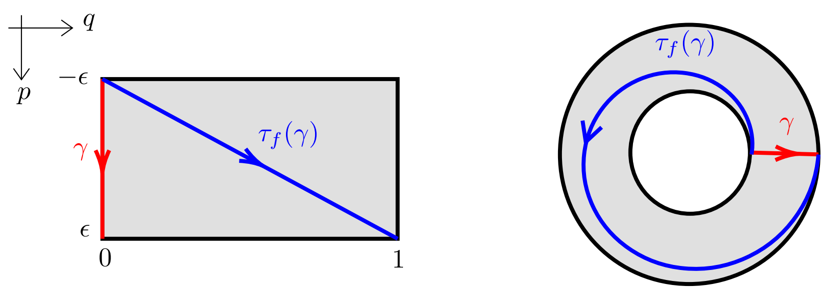


Figure 5: Image of τ_f.

Straightforward computations show that τ_f preserves the symplectic form $dp \wedge dq$, but does not preserve pdq.
Proof.

$$\tau_f^* pdq = pd(q + f(p)) = pdq + p\partial_f(p)dp,$$

Which implies

$$\tau_f * (dp \wedge dq) = dp \wedge dq.$$

Notice that $\tau_f^* pdq$ differs from pdq by the factor $p\partial_f(p)dp$. □
Let $f_\epsilon(p) = f(p/\epsilon)$. We'd like to construct a gluing that preserves the contact form, so that we may identify the contact forms inside and outside of $\nu_\epsilon(K)$ along its boundary after surgery. Hence, let

$$H_\epsilon(p) = \int_{-\infty}^{p} P\partial_p f_\epsilon(P)\, dP.$$

Roughly, H_ϵ measures the failure of τ_f in preserving pdq.
Let $0 < \delta < \epsilon/2$, and define regions T, B, S of an arc of $\nu_\epsilon(K)$ as in the figure below. The thickness of each component is given by δ.

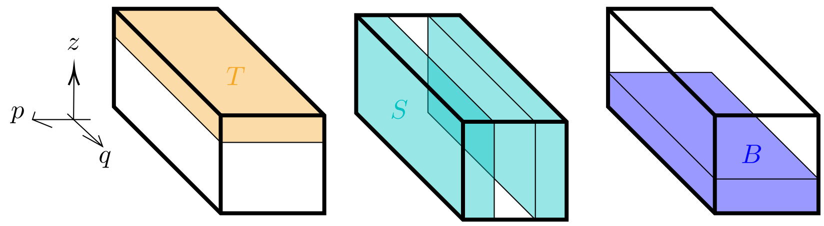


Figure 6: Gluing regions of $\nu_\epsilon(K)$.

Define the gluing map

$$\phi(z,p,q) \mapsto \begin{cases} (z + h_\epsilon(p), p, q - f_\epsilon(p)) & (z,p,q) \in T, \\ (z,p,q) & (z,p,q) \in B \cup S. \end{cases}$$

By taking δ to be sufficiently small compared to the smoothing of f_ϵ and H_ϵ, we can assure that the gluing agrees on the intersections $T \cap S$. We can easily visualize the effect of this surgery on the standard meridian:

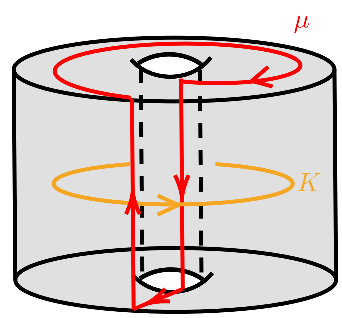


Figure 7: Image of μ after -1-surgery

From this, it is clear that the effect of this gluing is performing -1-surgery in the topological sense. Notice then that

$$\begin{aligned} \phi^*(dz + pdq) =& d(z + H_\epsilon(p)) + pd(q - f_\epsilon(p)) \\ =& dz + \partial_p H_\epsilon(p)dp + pdq - p\partial_p f_\epsilon(p)dp \\ =& dz + pdq + (\partial_p H_\epsilon(p) - p\partial_p f_\epsilon(p))dp \\ =& dz + pdq, \end{aligned}$$

where we use the fact that f_ϵ, and hence H_ϵ, is compactly supported. Thus, the gluing preserves the contact form. Notice that the Reeb vector field, viewed from outside the neighborhood, twists nontrivially around $\nu(K)$, rather than flowing vertically through the neighborhood as before.

2.3 Aternative Description

This description is used in [Gei08, Ch2, Ch4, Ch6], and in [Hon00]. We follow the conventions of [Gei08].

We now present a third description of Legendrian surgery, following the conventions of [Gei08]. This description uses a tubular neighborhood that is different from the one described in the previous section. The Legendrian push-offs in either case give a topologically different framing of the knot K, however the constructions achieve the same -1-surgered manifold.

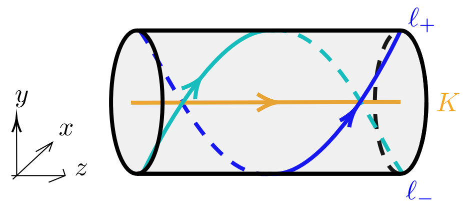
Let K be a Legendrian knot. Then there exists a tubular neighborhood of K, $\nu(K)$ contactomorphic to

$$(\mathbb{R}^2_{(x,y)} \times S^1_z, \lambda = \cos(z)dx - \sin(z)dy).$$

As before, $\{(0,0,z) \mid z \in S^1\}$ is identified with our knot in K. From the contact structure, we have two distinguished dividing curves on the boundary of the neighborhood, given by

$$\ell_\pm := \{(\pm\delta sin(z), \pm\delta\cos(z), z) \mid z \in S^1\}$$

We take $\ell_\pm$ to be the positive Legendrian push-off of K.



We begin by describing the effect of a general p/q-surgery on the contact form. Let (μ, ℓ_+) be a chosen meridian and longitude as above. Suppose we attempt the gluing given by

$$\mu \mapsto p\mu + q\ell_+ \qquad \ell_+ \mapsto s\mu + t\ell_+$$

Converting to polar coordinates $(x,y) \to (r,\theta)$,

$$\lambda = \cos(z+\theta)dr - r\sin(z+\theta)d\theta$$

Let $(\bar{r}, \alpha, \beta)$ be coordinates on the solid torus we'd like to glue back. Then the gluing gives transition maps

$$\theta = p\alpha + m\beta, \qquad \theta + z = q\alpha + n\beta.$$

And the pullback of the contact form along this gluing is thus

$$\bar{\lambda} = \cos(q\alpha + n\beta)dr - pr\sin(q\alpha+n\beta)d\alpha - mr\sin(q\alpha+n\beta)d\beta.$$

To determine whether this is still contact, it suffices to check that $\bar{\lambda} \wedge d\bar{\lambda}$ is a volume form. After a tedious calculation, we get that

$$\bar{\lambda} \wedge d\bar{\lambda} = r(pn - mq)d\alpha \wedge d\beta \wedge dr.$$

Since our gluing is an orientation-preserving diffeomorphism of $S^1 \times S^1$, we have that $pn - mq = 1 \neq 0$. So this is a well-defined volume form everywhere **except** when $r = 0$, i.e. along K.

Remark 3. We have a viable change of coordinates which, if we glue back along that identification, allows a thickened boundary of the neighborhood to inherit a contact structure.

Question. How can we extend the contact structure over the whole solid torus?

The answer was deduced by Honda [Hon00], using foundational work of Giroux. Giroux deduced that, given any convex surface in a contact manifold, its dividing set encodes all essential behaviour of the contact structure in a neighborhood of the surface. This is known as Giroux's Flexibility Theorem. This allows for the gluing of two contact manifolds along convex surfaces with the same dividing set.

Honda combined these results, along with work by Kanda and Etnyre to produce the following result enumerating the number of tight contact structures admitted by $S^1 \times D^2$:

Theorem 2.1 (Honda, 2000)**.** Consider the tight contact structures on $S^1 \times D^2$ with convex boundary T^2, for which $\#\Gamma = 2$ and $s(T^2) = -p/q$, $p \geq q > 0$, p and q coprime. Fix a characteristic foliation F which is adapted to Γ_{T^2} . There exist exactly $|(r_0+1)(r_1+1)...(r_{k-1}+1)(r_k)|$ tight contact structures on $S^1 \times D^2$ with this boundary condition, up to isotopy fixing T^2 . Here, $r0, , rk$ are the coefficients of the continued fraction expansion of $-p$:

$$-\frac{p}{q} = r_0 + \frac{1}{r_1 - \frac{1}{r_2 - \ldots r_k}}$$

Corollary 2.2 (Well-definedness when $p/q = 1/k$)**.** There is a unique tight contact structure on $S^1 \times D^2$ having convex boundary with these dividing curves.

Corollary 2.3. The construction above when $p/q = -1$ agrees with that of previous surgery descriptions. Indeed, the tubular neighborhoods of K in the handle description are related to the ones here by the contactomorphism sending $(q_1, q_2, \theta) \mapsto (-x, y, z)$.

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