HIRSCHI-WANG'S ANSWER TO DONALDSON'S 4-6 QUESTION

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These are notes from talk given at the Student Symplectic Summer Sessions Seminar at UC Berkeley, summer 2025. As is the nature of a seminar, thse accompanying notes are very rough and may contain typos, aproximations, and errors. If any are found, fee free to email me with any corrections.

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1. Introduction

Theorem 1.1 (Wall, 1964)**.** If two simply-connected 4-manifolds X and Y are homeomorphic, then $X \times S^2$ and $Y \times S^2$ are diffeomorphic.

Question. Does a similar result hold in the symplectic setting?

This question is attributed to Donaldson as problem 5 and has been an open problem for 25 years.

Definition 1.2. Two symplectic manifolds (X_0, ω_0) and (X_1, ω_1) are *deformation equivalent* if there exists a diffeomorphism $\psi : X_0 \to X_1$ such that ψ^* is homotopic to ω via a path of symplectic forms.

Question (Donaldson's 4-6)**.** Suppose (X_0, ω_0) and (X_1, ω_1) are two closed, simply connected symplectic 4-manifolds. Then the following are equivalent:

(1) $X_0 \simeq_{\text{diff}} X_1$,
(2) $(X_0, \omega_0) \times (S^2, \omega_{\text{std}}) \simeq_{\text{def eq.}} (X_1, \omega_1) \times (S^2, \omega_{\text{std}})$.

The procedure by which we take a product with $(S^2, \omega_{\text{std}})$ is called *stabilization.* This question appeared as a conjecture for the first time in Ruan and Tian's paper in 1997. It became known as the *stabilizing conjecture.*

Theorem 1.3 (Ruan-Tian, 1997)**.** The stabilizing conjecture is true for simply connected rational elliptic surfaces.

More supporting evidence for this conjecture is given by Ionel-Parker (1999), and YonSeung Cho (2014). However, it was known early on that there exists symplectic 4-manifolds with non-deformation equivalent symplectic forms. The question whether this behaviour could persist after stabilization was not known though until the following result of Hirschi-Wang in 2024:

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Theorem 1.4 (Hirschi-Wang, 2024)**.** There exist infinitely many examples of smooth, simply-connected 4-manifolds admitting symplectic forms that remain inequivalent under stabilization.

More precisely, there exist infinitely many pairwise nonhomeomorphic smooth closed simply-connected 4-manifolds X admitting symplectic forms ω_0 and ω_1 so that the product forms $\omega_0 \oplus (\omega_{\text{std}})^{\oplus k}$ and $\omega_1 \oplus (\omega_{\text{std}})^{\oplus k}$ are deformation inequivalent for any $k \geqslant 1$.
In other words, the result of Hirschi and Wang gives a counterexample to the conjecture (1) $\Longrightarrow$ (2). In the other direction, they improved previously known results to the following:

Theorem 1.5. Let (X_0, ω_0) and (X_1, ω_1) be two closed, simply-connected symplectic manifolds so that $(X_0, \omega_0) \times (S^2, \omega_{\text{std}}) \simeq_{\text{def eq.}} (X_1, \omega_1) \times (S^2, \omega_{\text{std}})$. Then the Gromov-Witten invariants of (X_0, ω_0) agree with those of (X_1, ω_1) up to homeomorphism.

As a corollary of this, we have that

Corollary 1.6. If (X_0, ω_0) and (X_1, ω_1) are as in the theorem above, and $b_2^+(X_i) \geqslant 2$, then their Seiberg-Witten invariants are intertwined by a homeomorphism.

Remark 1. Both of the above results hold when replacing $(S^2, \omega_{\text{std}})$ with $(S^2, (\omega_{\text{std}})^{\oplus k})$

As a consequence of this remark, we have that

Corollary 1.7. In any dimension $\geqslant 6$, there exist infinitely many smooth manifolds admitting inequivalent symplectic forms.

The goal of this talk is to explain the first theorem that we stated by Hirschi-Wang. Time permitting, we will discuss other aspects of their paper.

2. Background

The first question we should ask is why think about stabilization in the first place. Well, its certainly a natural construction that gives compact higher dimensional manifolds in whichever category you like. Off of this, a natural question to ask is how much of the original manifold's structure is retained after stabilizing? This question has been explored extensively in the geometric topology setting. For example, it is known that any exotic structure on a closed smooth manifold vanishes (i.e. is no longer exotic) under stabilization.
The second question we may ask is why is deformation equivalence the right notion to think about equivalence of symplectic manifolds under stabilization?
Finally, before diving into the proof of the theorem, we need to introduce an important definition that will help us make sense of deformation equivalence.

Definition 2.1 (Cohomology equivalence)**.** A continuous map $f : X \to X$ is called a *cohomology equivalence* if it induces an isomorphism f^* on singular cohomology with integral coefficients.

Definition 2.2 (Set of cohomology equivalences)**.** Denote by $G_{X,Y}$ the set of cohomology equivalences ψ of $X \times Y$ satisfying:

(1) ψ preserves $H^2(X; \mathbb{Z})$ setwise,
(2) $pr_1 \circ \psi(\cdot, y)$ is a cohomology equivalence on X for all $y \in Y$.

Note that in order to make sense of this, we'll require that Y is path-connected. Then we may treat $H^2(X; \mathbb{Z})$ as a subspace of $H^2(X \times Y; \mathbb{Z})$ considered as a vector space.
The last question we may ask ourselves is why is this notion important?

3. Main Result

The key theorem used in proving theorem 1.4 is the following:

Theorem 3.1. Let Σ be a closed surface with a symplectic form σ and X a simply-connected 4-manifold with nonvanishing signature. If ω_0 and ω_1 are two symplectic forms on X with $c_1(\omega_0)$ and $c_1(\omega_1)$ in different *orbits of action of cohomology equivalences* of X on $H^2(X;\mathbb{Z})$, then $\omega_0 \oplus \sigma^{\oplus k}$ and $\omega_1 \oplus \sigma^{\oplus k}$ on $X \times \Sigma^k$ are deformation inequivalent for any $k \geqslant 1$.

Proof. We'll stick to $k = 1$ for ease of notation, but this easily extends to $k \geqslant 2$. Suppose for contradiction's sake that the above initial assumptions hold, but that $\omega_0 \oplus \sigma$ and $\omega_1 \oplus \sigma$ on $X \times \Sigma$ are deformation equivalent. In particular, we assume $\sigma(X) \neq 0$. Let $\psi : X \times \Sigma \to X \times \Sigma$ be a diffeomorphism arising from the deformation equivalence.
Naturality of the Chern classes tells us that

$$c_1(\psi^*(\omega_1 \oplus \sigma)) = \psi^* c_1(\omega_1 \oplus \sigma).$$

Moreover, by the fact that $\psi^*(\omega_1 \oplus \sigma)$ is path connected to $\omega_0 \oplus \sigma$, we have that

$$c_1(\omega_0 \oplus \sigma) = \psi^* c_1(\omega_1 \oplus \sigma).$$

We gather then that the first Chern classes of $\omega_0 \oplus \sigma$ and $\omega_1 \oplus \sigma$ are related by the diffeomorphism ψ. One can then show that for any $\psi \in \mathrm{Diff}(X \times \Sigma)$, $\psi \in G_{X,\Sigma}$. Knowing this would then tell us that $c_1(\omega_0 \oplus \sigma)$ and $c_1(\omega_1 \oplus \sigma)$ live in the same orbit of action of $G_{X,\Sigma} \curvearrowright H^*(X \times \Sigma;\mathbb{Z})$. But this is not true by the following proposition.

Proposition 3.2. Let X be a closed, simply connected 4-manifold with $\sigma(X) \neq 0$. Suppose also that ω_0 and ω_1 are symplectic form on X such that $c_1(\omega_0)$ and $c_1(\omega_1)$ lie in different orbits of action of cohomology equivalences of X on $H^2(X;\mathbb{Z})$. Then the first Chern classes $c_1(\omega_0 \oplus \sigma)$ and $c_1(\omega_1 \oplus \sigma)$ must also lie in different orbits of action of $G_{X,\Sigma}$.

In other words, if we can show that for any $\psi \in \mathrm{Diff}(X \times \Sigma)$, $\psi \in G_{X,\Sigma}$, then we will have a contradiction. Our initial assumption will have to be wrong, so $(X \times \Sigma, \omega_0 \oplus \sigma)$ will be deformation inequivalent to $(X \times \Sigma, \omega_1 \times \sigma)$.
So let's prove that $\psi \in G_{X,\Sigma}$. Recall that $\psi \in G_{X,\Sigma}$ if ψ maps $H^2(X;\mathbb{Z})$ to itself set-wise, and $pr_1\psi(\cdot, p)$ is a cohomology equivalence on X for all $p \in \Sigma$.
Let's start by showing that ψ preserves $H^2(X;\mathbb{Z})$. Since Σ is a surface, $p_1(\Sigma)$ vanishes - remember it lives in the fourth cohomology group of Σ. Moreover, $H^*(X \times \Sigma;\mathbb{Z})$ is torsion-free. Since Pontryagin classes satisfy the Whitney formula up to 2-torsion, it follows that

$$\begin{aligned}
p_1(X \times \Sigma) &= p_1(pr_1^*(TX) \oplus pr_2^*(T\Sigma)) \\
&= p_1(pr_1^*(TX)) \cup p_1(pr_2^*(T\Sigma)) \\
&= pr_1^* p_1(X) \\
&= pr_1^*(3\sigma(X)) \\
&= 3\sigma(X) PD([\Sigma])
\end{aligned}$$

The equalities are due to (1) the definition of the tangent space of a product manifold, (2) the Whitney sum formula for Pontryagin classes, (3) naturality of Pontryagin classes, (4) the Hirzebruch signature theorem for 4-manifolds.
Now, since diffeomorphisms preserve p_1 and $H_*(X \times \Sigma;\mathbb{Z})$ is gree, $\psi_*[\Sigma] = \pm[\Sigma]$. Let $\alpha \in H^2(X;\mathbb{Z})$. We can think of $H^2(;\mathbb{Z}) \subseteq H^2(X \times \Sigma;\mathbb{Z})$ as a subspace, so that $\psi^*\alpha$ makes sense. We'd like to

show that $H^2(X;\mathbb{Z})$ is preserved by ψ. A priori, $\psi^*\alpha = \alpha' + ah$, where h is the algebraic dual of $[\Sigma]$. On one hand,

$$\alpha \cap [\Sigma] = 0$$

where again we think of $\alpha \in H^2(;\mathbb{Z}) \subseteq H^2(X \times \Sigma;\mathbb{Z})$. On the other hand,

$$\begin{aligned} \pm\alpha \cap [\Sigma] &= \alpha \cap \psi_*[\Sigma] \\ &= \psi_*(\psi^*(\alpha) \cap [\Sigma]) \\ &= \psi_*((\alpha' + ah) \cap [\Sigma]) \\ &= \psi_*(ah[\Sigma]) \end{aligned}$$

So that $a = 0$. The equalities are due to (1) the fact that $\psi_*[\Sigma] = \pm[\Sigma]$, (2) a standard identity for the cap product, (3) by observation, and (4) by pairing.
In other words, we have shown that for any $\alpha \in H^2(X;\mathbb{Z})$, $\psi^*\alpha \in H^2(X;\mathbb{Z})$. So ψ^* preserves $H^2(X;\mathbb{Z})$.
Now, let's show the second part: that $pr_1\psi(\cdot, p)$ is a cohomology equivalence on X for each $p \in \Sigma$. We define $\widehat{\psi} : X \to X$ by $\widehat{\psi}(x) = pr_1(x, p)$ for some arbitrary $p \in \Sigma$. Since $\psi_*[\Sigma] = \pm[\Sigma]$, $\widehat{\psi}$ must have degree ± 1. To see why this is, note that ψ is a diffeomorphism, and hence sense $\psi_*([X \times \Sigma] \to \pm[X \times \Sigma]$, depending on whether ψ is orientation preserving or reversing. On the other hand, we know that both homologies of X and Σ are free $\mathbb{Z}$-modules, and so the Künneth theorem says that $[X \times \Sigma] \simeq [X] \oplus [\Sigma]$. So under this we have
I.e., $(pr_1 \circ \psi)_*[X] = \pm[X]$. In other words, $\widehat{\psi}$ has degree ± 1. Thus, for any $\alpha, \beta \in H^2(X;\mathbb{Z})$

$$\widehat{\psi}^*\alpha \cup \widehat{\psi}^*\beta = \widehat{\psi}^*(\alpha \cup \beta) = \pm\alpha \cup \beta.$$

So $\widehat{\psi}$ preserves the cup product, and hence preserves $H^2(X;\mathbb{Z})$ as a ring???? This establishes that $\widehat{\psi}$ induces an isomorphism on cohomology - i.e. is a cohomology equivalence. □
For sake of time, we're not going to prove proposition 3.2, but it can be shown using a similar idea to the above.
We now use theorem 3.1 to prove theorem 1.4.
Proof.
Work of Smith in 2000 provides candidates that satisfy the conditions of theorem 3.1. This is all that we need to show, then we will have theorem 1.4
Smith's argument constructs candidates roughly as follows. Start with T^4 along with the standard contact structure (?). He shows that you can perturb the symplectic structure, and then take the symplectic fibre sum of the resulting symplectic manifold with $(n+3)$ copies of the rational elliptic surface $E(1)$. The perturbation only happens at the level of the symplectic structure, and leaves the diffeomorphism type of the underlying manifold Z_n the same. However, we end up with n symplectic structures on Z_n whose Chern classes have coprime divisibilities. Moreover, if $m \neq n$, then Z_m is not homeomorphic to Z_n. So we get an infinite family of topologically distinct manifolds that each carry sets of deformation inequivalent symplectic structures.
In particular, the fact that the Chern classes of the resulting symplectic structures have coprime divisabilities means that they must lie in distinct orbits of action of cohomology equivalences. All that is left to show is that the underlying manifold Z_n has nonvanishing signature. Then we will have satisfied all the conditions of theorem 3.1.
Let's unpack the construction of Z_n. It is the fibre sum of T^4 with $(n+3)$ copies of $E(1)$. In particular, we can set up the gluing to be by an orientation preserving diffeomorphism of the

attaching regions, so that the fibre sum gives us additivity of the signatures of its constituent manifolds:

$$\sigma(Z_n) = \sigma(T^4) + (n+3)\sigma(E(1)).$$

The Pontryagin classes behave similarly:

$$p_1(Z_n) = p_1(T^4) + (n+3)p_1(E(1)).$$

Now, T^4 is a Riemann flat manifold, and so Chern-Weil theory tells us that $p_1(T^4) = 0$. On the other hand, $\sigma(E(1)) = -8$, and so the Hirzebruch signature theorem tells us that $p_1(E(1) = -24$. Hence,

$$\langle p_1(Z_n), [Z_n] \rangle = (n+3)(-24) \neq 0.$$

A second application of Hirzebruch signature theorem says then that $\sigma(Z_n) \neq 0$. We are done. $\square$

4. Further aspects

There are other parts of this exposition can still be addressed:

(1) We can replace (Σ, σ) with $(\Sigma^k, \sigma^{\oplus k})$ for any $k \geqslant 1$.
(2) Hirschi and Wang also make use of a second family of examples, which we could talk about. In particular, they look at stabilizing with $(\mathbb{C}P^k, \omega_{FS}^{\oplus k})$, and prove the same result of theorem 1.4 in this setting.
(3) The reverse direction of the stabilizing conjecture is more subtle, and we could talk about the product structure on Gromov Witten invariants that addresses this direction.

References

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