Discussion Worksheet
Math 53 Worksheets for Fall 2023[1]

MATH 53 Multivariable Calculus

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Contents

[1]These worksheets are provided for the personal use of Fall 2023 MATH 53 DIS 106/109 students only.

1 Introduction - 8.23

Discussion Site : https://math.berkeley.edu/~ning_tang/math53_F23/index.html

1.1 Topics you will learn :

(1) polar coordinates;

(2) how to express curves, surfaces;

(3) chain rule, partial derivative;

(4) Lagrange multipliers, maxima and minima;

(5) double integrals, iterated integrals, triple integrals;

(6) line integrals, surface integrals;

(7) Green's theorem, Stokes's Theorem, divergence theorem.

1.2 What you have learn correspondingly :

(1) Catesian coordinates;

(2) how to express a line;

(3) derivatives of a single variable function;

(4) how to find the maximum/minimum of a single variable function;

(5) (in)definite integrals, improper integrals;

(6) how to calculate the surface area of a surface of revolution;

(7) integration by parts.

1.3 Trigonometric identities

a) $\sin^2 x + \cos^2 x = 1$;

b) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$;

c) $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$;

d) $\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$,
$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$,
$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$.

1.4 Arc Length Formula and Surface of Revolution

If $f(x)$ is a function on $[a,b]$ such that $f'(x)$ is continuous on $[a,b]$, then the arc length L of the curve $y = f(x)$ (or $x = g(y)$ as y goes from c to d, after inverting) is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_c^d \sqrt{1 + [g'(y)]^2} dy,$$

and the surface area S of the solid from revolving $y = f(x)$ as x goes from a to b (or, again, $x = g(y)$ as y goes from c to d) about the x-axis is given by

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} dx = 2\pi \int_c^d y\sqrt{1 + [g'(y)]^2} dy.$$

The surface area obtained by revolving the same curve around the y-axis is given by

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2} dx = 2\pi \int_c^d g(y)\sqrt{1 + [g'(y)]^2}.$$

If $f'(x)$ or $g'(y)$ is discontinuous or undefined somewhere on $[a,b]$ (including the ends a or b), the above integrals should be treated as indefinite, and the standard modifications of the integrals via limits must be applied.

1.5 A list of basic integrations you should grasp

a) $\int c\,dx = cx + C$

b) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (where $n + 1 \neq 0$)

c) $\int \frac{1}{x}\,dx = \ln|x| + C$

d) $\int e^x\,dx = e^x + C$

e) $\int a^x\,dx = \frac{a^x}{\ln a} + C$

f) $\int \sin x\,dx = -\cos x + C$

g) $\int \cos x\,dx = \sin x + C$

h) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

i) $\int \frac{1}{\cos^2 x}\,dx = \tan x + C$

j) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$

k) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C$

Substitution rule :

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad (u = g(x),\ du = u'(x)\,dx)$$

Integration by parts :

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

2 Parametric equations - 8.25

1 (1) Eliminate the parameter for the parametric equations to find a Cartesian equation of the curve; (2) Sketch the graph of the parametric curve and indicate with an arrow the direction in which the curve is traced as the parameter increases :

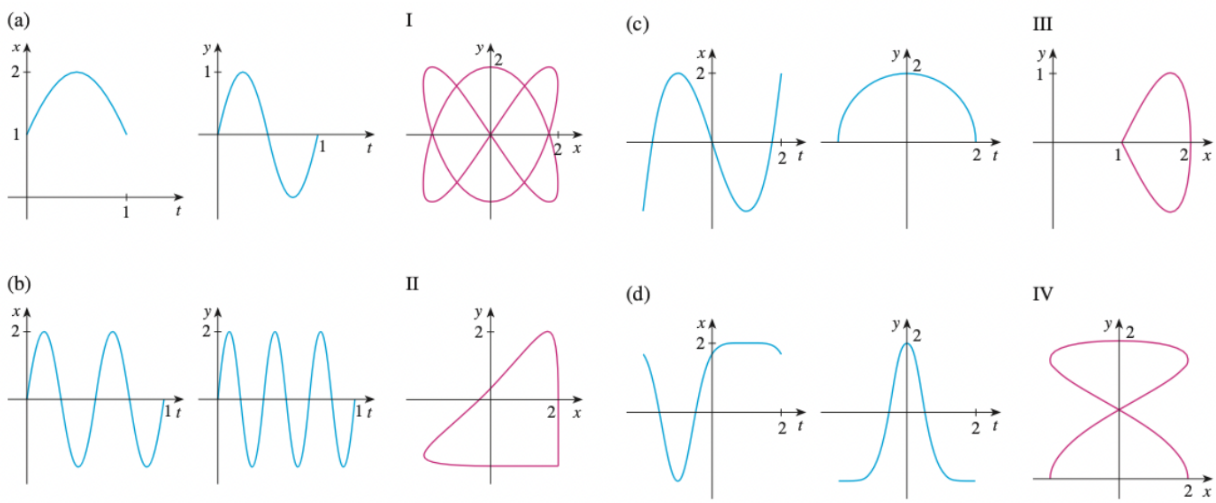
- [T, §11.1, #1] $x = 3t$, $y = 9t^2$, $-\infty < t < \infty$;
- [T, §11.1, #5] $x = \cos(2t)$, $y = \sin(2t)$, $0 \le t \le 2\pi$;
- [T, §11.1, #7] $x = 4\cos t, y = 2\sin t$, $0 \le t \le 2\pi$;
- [T, §11.1, #13] $x = t$, $y = \sqrt{1-t^2}$, $-1 \le t \le 0$.

2 [§10.1, #21] Describe the motion of a particle with position

$$x = 5\sin t,\ y = 2\cos t,\ -\pi \le t \le 5\pi$$

as t varies in the given interval.

3 [§10.1, #24] Match the graphs of the parametric equations $x = f(t)$ and $y = g(t)$ in $(a) - (d)$ with the parametric curves labeled I–IV. Give reasons for your choices.



4 [T, §11.1, #19] Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$

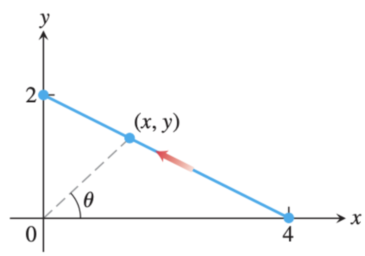
- once clockwise.
- once counterclockwise.
- twice clockwise.
- twice counterclockwise.

(There are many ways to do these, so your answers may not be the same.)

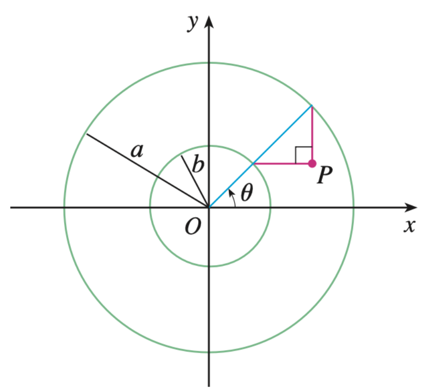
3 Parametric equations, Tangents and Area - 8.28

1 [T, §11.1, #21] Find a parametrization for the line segment with endpoints $(-1, -3)$ and $(4, 1)$.

2 [T, §11.1, #31] Find a parametrization for the line segment joining points (0, 2) and (4, 0) using the angle u in the accompanying figure as the parameter.



3 [§10.1, #41] If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. Then eliminate the parameter and identify the curve.



4 [§10.2, Example 2] Recall the cycloid obtained by rolling a circle of radius r:

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta)$$

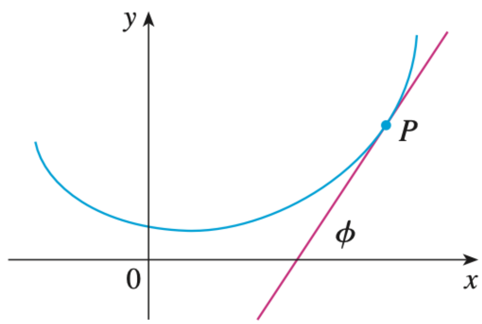
Find the slope of the tangent line in terms of θ. Where are the tangents horizontal and vertical?

5 Use the arc length formula for the following parametric equation : $x = 3\sin 3t$, $y = 3\cos 3t$, $0 \leq t \leq 2\pi$.

6 Find the arc length of the curve $x(t) = e^t + e^{-t}, y(t) = 2t - 5$ for $0 \leq t \leq 3$.

7 Find the area of the region enclosed by the curve $x(t) = 1 - t, y(t) = e^t$ and the vertical lines $x = 0$, $x = 2$.

8 [§10.2, #69] The curvature at a point P of a curve is defined as $\kappa = |\frac{d\phi}{ds}|$ where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.



a) For a parametric curve $x = x(t), y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

where the dots indicate derivatives with respect to t. [Hint: Use $\phi = \arctan(dy/dx)$ to find $d\phi/dt$ and use chain rule to find $d\phi/ds$.]

b) By regarding a curve $y = f(x)$ as the parametric curve $x = x, y = f(x)$ with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{(1 + (dy/dx)^2)^{\frac{3}{2}}}.$$

4 Polar coordinates - 8.30

1 Sketch the polar curve $r = \theta$. What is it in Cartesian coordinates $(x(\theta), y(\theta))$?

2 Sketch the polar curve $\theta = \pi/4$ and write its defining equation in x and y.

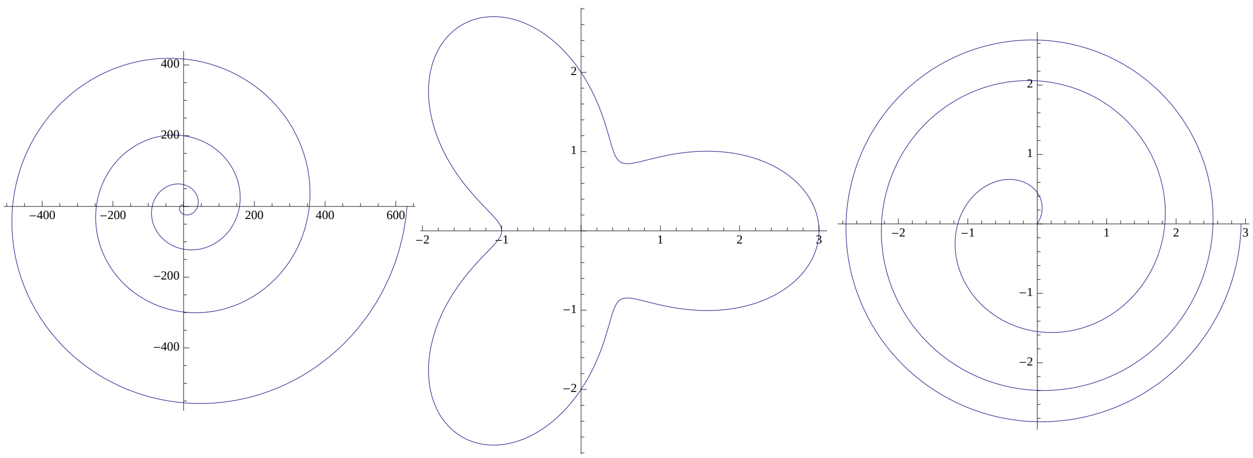
3 Sketch the polar curve $r = \sin 3\theta$.

4 [§10.3, #54] Match the polar curve equations with their corresponding pictures.

a) $r = \ln\theta, \quad 1 \le \theta \le 6\pi$;

b) $r = \theta^2, \quad 0 \le \theta \le 8\pi$;

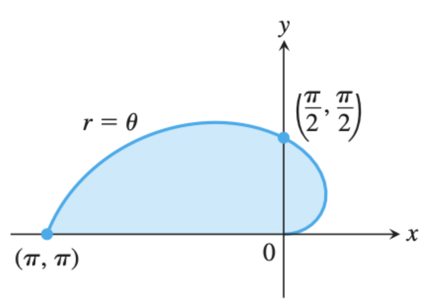
c) $r = 2 + \cos 3\theta$.



5 [§10.4, #31] Find the area of the region that lies inside both curves $r = \sin 2\theta$ and $r = \cos 2\theta$.

6 [§10.4, #41] Find all points of intersection of the two curves $r_1 = \sin\theta$ and $r_2 = \sin 2\theta$.

7 [T, §11.5, #1] Find the area of the region bounded by $r = \theta$ for $0 \le \theta \le \pi$.



8 [T, §11.5, #5] Find the area of the region inside one leaf of the four-leaved rose $r = \cos 2\theta$.

9 [T, §11.5, #9] Find the areas of the regions shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$.

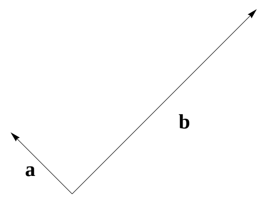
10 [T, §11.5, #21] Find the lengths of the spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$.

11 [T, §11.5, #23] Find the lengths of the curve $r = 1 + \cos\theta$.

12 [T, §11.5, #25] Find the lengths of the parabolic segment $r = \dfrac{6}{1+\cos\theta}$, $0 \le \theta \le \pi/2$.

5 Vectors, Dot product - 9.1

1 Draw $\overrightarrow{a} + \frac{1}{2}\overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$ given the vectors below.



2 Find a vector in the direction of $\langle -2, 4, 2\rangle$ and of length 6.

3 If $\overrightarrow{v}$ lies in the first quadrant and makes an angle of $\pi/3$ with the positive x-axis and $|\overrightarrow{v}| = 4$, find $\overrightarrow{v}$ in component form.

4 Prove that the diagonals of a parallelogram intersect at their midpoints. ("Prove" just means explain why it's true.)

5 Consider the three points $A(1, 1, 1)$, $B(1, 0, 1)$ and $C(1, 0, 0)$.

(1) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in components.

(2) Find the vector lengths $|\overrightarrow{AB}|$ and $|\overrightarrow{AC}|$.

(3) Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

(4) Find the angle between these two vectors using the dot product.

(5) Verify your answer by plotting the points and drawing a picture.

6 Find the angle between a diagonal of a cube and a diagonal of one of its faces.

7 Find the scalar and vector projections of $\overrightarrow{b} = \langle 0, 1, \frac{1}{2}\rangle$ onto $\overrightarrow{a} = \langle 2, -1, 4\rangle$.

6 Cross product - 9.6

1 [§12.4, #29] a) Find a nonzero vector orthogonal to the plane through the points $P(1, 0, 1)$, $Q(-2, 1, 3)$, $R(4, 2, 5)$ and b) find the area of triangle PQR.

2 Prove the addition trig formulas for $\cos(\theta_2 - \theta_1)$ and $\sin(\theta_2 - \theta_1)$ using the dot product and cross product. (Start by taking unit vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ at angles θ_1 and θ_2.)

3 Find $\langle 6, 0, -2\rangle \times \langle 0, 8, 0\rangle$.

4 [T, §12.4, #1] Find $\overrightarrow{u} \times \overrightarrow{v}$ and $\overrightarrow{v} \times \overrightarrow{u}$ for $\overrightarrow{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{v} = \mathbf{i} - \mathbf{k}$.

5 Geometrically, why is $(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{a} = 0$ for all vectors $\overrightarrow{a}$, $\overrightarrow{b}$ in $\mathbb{R}^3$?

6 Find two unit vectors orthogonal to both $\overrightarrow{a} = \langle 1, 2, 1\rangle$ and $\overrightarrow{b} = \langle -3, 1, 0\rangle$. What other vectors are orthogonal to both $\overrightarrow{a}$ and $\overrightarrow{b}$?

7 [T, §12.4, #35] Find the area of the parallelogram with vertices $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$.

8 [T, §12.4, #37] Find the area of the parallelogram with vertices $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$ and $D(4, 3)$.

9 Given $P(1, 2, 3)$, $Q(1, 3, 6)$, $R(3, 5, 6)$ and $S(1, 4, 2)$.

a) Find the area of the triangle with vertices P, Q and R.

b) Find $\overrightarrow{PS}$.

c) Find the volume of the parallelepiped spanned by $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.

7 Lines and planes - 9.8

1 Find the parametric and symmetric equations for the line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$.

2 Find the equation of a plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

8 Lines and planes - 9.11

1 [T, §12.5, #13] Find a parametric form of the line joining the points

$$(0,0,0), \quad (1,1,3/2).$$

2 Find the equation of the plane through $P_0(2,4,5)$ perpendicular to the line $x = 5+t, y = 1+3t, z = 4t$.

3 [T, §12.5, #33, #35] Find the distance from the point to the line :

a) $(0,0,12)$, $x = 4t,\ y = -2t,\ z = 2t$;

b) $(2,1,3)$, $x = 2+2t,\ y = 1+6t,\ z = 3$.

4 [T, §12.5, #39, #41] Find the distance from the point to the plane :

a) $(2,-3,4), \quad x+2y+2z = 13$;

b) $(0,1,1), \quad 4y+3z = -12$

5 Find the equation of the plane containing the following point and line: the point is $(-1,2,1)$ and the line is given by the intersection of the planes $x+y-z = 2$ and $2x-y+3z = 1$.

6 [T, §12.5, #29] Find the plane containing the intersecting lines

$$L1 : x = -1+t, \quad y = 2+t, \quad z = 1-t; \quad -\infty < t < \infty$$
$$L2 : x = 1-4s, \quad y = 1+2s, \quad z = 2-2s; \quad -\infty < s < \infty$$

7 [T, §12.5, #31] Find a plane through $P_0(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x+y-z = 3, x+2y+z = 2$.

9 Vector functions, Space curves - 9.13

1 If $\overrightarrow{u}(t) = \langle \sin t, \cos t, t\rangle$ and $\overrightarrow{v}(t) = \langle t, \cos t, \sin t\rangle$, find $\frac{d(\overrightarrow{u} \cdot \overrightarrow{v})}{dt}$ and verify that it equals $\frac{d\overrightarrow{u}}{dt} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \frac{d\overrightarrow{v}}{dt}$.

2 Find the tangent vector $\overrightarrow{r}'(t)$ for the curve $\overrightarrow{r}(t) = \langle t^2, \cos 2t, -te^t\rangle$. Find the equation of the tangent line to the curve at $t = 0$.

3 Find the velocity, acceleration, and speed of a particle with the position function $\overrightarrow{r}(t) = e^t(\cos t\ \hat{\mathbf{i}} + \sin t\ \hat{\mathbf{j}} + t\ \hat{\mathbf{k}})$.

4 Find the velocity and position vectors of a particle that has acceleration $\overrightarrow{a}(t) = \hat{\mathbf{i}} + 2\ \hat{\mathbf{j}}$, $\overrightarrow{v}(0) = \hat{\mathbf{k}}$, $\overrightarrow{r}(0) = \hat{\mathbf{i}}$.

5 The position function of a spaceship is

$$\overrightarrow{r}(t) = (3 + t)\ \hat{\mathbf{i}} + (2 + \ln t)\ \hat{\mathbf{j}} + \left(7 - \frac{4}{t^2 + 1}\right)\ \hat{\mathbf{k}}$$

and the coordinates of a space station are $(6, 4, 9)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off?

6 [T, §13.2, #1] Evaluate the integral

$$\int_0^1 \left[t^3\mathbf{i} + 7\mathbf{j} + (t + 1)\mathbf{k}\right]\ dt$$

7 [T, §13.1, #15] Let $r(t)$ be the position of a particle in space at time t.

$$\mathbf{r}(t) = (3t + 1)\mathbf{i} + \sqrt{3}t\mathbf{j} + t^2\mathbf{k}$$

Find the angle between the velocity and acceleration vectors at time $t = 0$.

10 Two variable functions, Tangent planes - 9.18

1 [T, §14.1, #5] Find the domain of $f(x, y) = \sqrt{y - x - 2}$.

2 [T, §14.1, #9] Find the domain of $f(x, y) = \cos^{-1}(y - x^2)$.

3 [T, §14.2, #13] Find the limits :

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - 2xy + y^2}{x - y}$$

4 [T, §14.2, #15] Find the limit :

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy - y - 2x + 2}{x - 1}$$

5 [T, §14.3, #43] Calculate all the second derivatives of

$$g(x, y) = x2y + cosy + ysinx.$$

6 [T, §14.3, #56] Calculate

$$\frac{\partial^5 f}{\partial x^2 \partial y^3}$$

with

$$f(x, y) = y^2 + y\left(\sin x - x^4\right).$$

7 [T, §14.6, #1] Find the tangent plane to the surface

$$x^2 + y^2 + z^2 = 3, \quad P_0(1, 1, 1).$$

8 [T, §14.6, #5] Find the tangent plane to the surface

$$\cos \pi x - x^2 y + e^{xz} + yz = 4, \quad P_0(0, 1, 2).$$

9 [T, §14.6, #9] Find an equation for the plane that is tangent to the given surface at the given point. $z = \ln(x^2 + y^2)$, $(1, 0, 0)$.

10 [T, §14.3, #72] Let $f(x, y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq 0, \\ 0, & \text{if } (x, y) = 0. \end{cases}$

a) Show that $\frac{\partial f}{\partial y}(x, 0) = x$ for all x, and $\frac{\partial f}{\partial x}(0, y) = -y$ for all y.

b) Show that $\frac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \frac{\partial^2 f}{\partial x \partial y}(0, 0)$.

11 Continuity of functions, Limits, Differentials - 9.20

1 [T, §14.6, #25] Find the linearization $L(x, y)$ of the function $f(x, y) = x^2 + y^2 + 1$ at each point :

- $(0, 0)$,
- $(1, 1)$.

2 [T, §14.6, #57] A smooth curve is tangent to the surface at a point of intersection if its velocity vector is orthogonal to ∇f there. Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t - 1)\mathbf{k}$$

is tangent to the surface $x^2 + y^2 - z = 1$ when $t = 1$.

3 [T, §14.2, #31] Determine at which points, the functions are continuous :

- $f(x, y) = \sin(x + y)$;
- $f(x, y) = \ln(x^2 + y^2)$

4 By considering different paths of approach, show that the functions have no limit as $(x, y) \to (0, 0)$:

a) [T, §14.2, #41] $f(x, y) = -\frac{x}{\sqrt{x^2+y^2}}$;

b) [T, §14.2, #43] $f(x, y) = \frac{x^4-y^2}{x^4+y^2}$.

5 Find an equation for and sketch the graph of the level curve of the function $f(x, y)$ that passes through the given point.

(1) [T, §14.1, #49] $f(x, y) = 16 - x^2 - y^2, \quad (2\sqrt{2}, \sqrt{2})$;

(2) [T, §14.1, #51] $f(x, y) = \sqrt{x + y^2 - 3}, \quad (3, -1)$.

12 Chain Rule, Implicit Differentiation, Gradients - 9.28

1 Find $\dfrac{dz}{dt}$ where $z = \sqrt{1 + x^2 + y^2}$ and $x = \ln t$, $y = \cos t$.

2 Let $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$. Use the Chain Rule to find $\partial w/\partial r$ and $\partial w/\partial\theta$ at $(r, \theta) = (2, \pi/2)$.

3 Use implicit differentiation to find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ where $e^z = xyz$.

4 [T, §14.4, #25] Assuming the equation $x^3 - 2y^2 + xy = 0$, use implicit differentiation to compute dy/dx at the point $(1, 1)$.

5 [T, §14.4, #29] Find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points $(1, 1, 1)$, where $z^3 - xy + yz + y^3 - 2 = 0$.

6 [T, §14.4, #45] Laplace equations : Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

7 [T, §14.5, #7] Find ∇f at the given point :

$$f(x, y, z) = x^2 + y^2 - 2z^2 + z\ln x, \quad (1, 1, 1).$$

8 [T, §14.5, #11] Find the derivative of the function at P_0 in the direction of $\mathbf{u}$:

$$f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$$

9 Find ∇f for $f(x, y, z) = x^2yz - xyz^3$. Evaluate the gradient at the point $P(2, -1, 1)$. Find the rate of change of f at P in the direction of the vector $\overrightarrow{u} = \langle 0, 4/5, -3/5\rangle$.

10 Show that $\nabla(uv) = u\nabla v + v\nabla u$ for u, v differentiable functions of x and y.

11 Suppose that over a certain region of space the electric potential is given by $V(x, y, z) = 5x^2 - 3xy + xyz$. Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\overrightarrow{v} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} - \hat{k}\right)$.

13 Maxima and minima - 10.2

1 Find the local maximum and minimum values and saddle points of the following functions:

a) [T, §14.7, #1] $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$;

b) [T, §14.7, #23] $f(x, y) = y \sin x$;

c) [T, §14.7, #25] $f(x, y) = e^{x^2+y^2-4x}$;

d) [T, §14.7, #29] $f(x, y) = 2 \ln x + \ln y - 4x - y$.

2 [T, §14.7, #43] Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that

a) $f_x = 2x - 4y$ and $f_y = 2y - 4x$;

b) $f_x = 2x - 2$ and $f_y = 2y - 4$;

c) $f_x = 9x^2 - 9$ and $f_y = 2y + 4$.

Describe your reasoning in each case.

3 Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$. (Hint : First, you want to find what we want to minimize : The distance d of a point (x, y, z) from the point $P(2, 0, -3)$ is given by $d^2 = (x - 2)^2 + y^2 + (z + 3)^2$.)

4 [T, §14.7, #51] Find the point on the plane $3x + 2y + z = 6$ that is nearest the origin.

14 Global max/min, Lagrange multipliers - 10.6

1 Find the absolute maximum and minimum values of $f(x,y) = 2x^3 + y^4$ on the domain $\{(x,y) \mid x^2 + y^2 \leq 1\}$.

2 [T, §14.7, #31] Find the absolute maxima and minima of the functions on the given domains : $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant.

3 [T, §14.8, #45] The condition $\nabla \boldsymbol{f} = \lambda \nabla \boldsymbol{g}$ is not sufficient : Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x,y)$ subject to the conditions $g(x,y) = 0$ and $\nabla g \neq \mathbf{0}$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x,y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4,4)$ and $(-4,-4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x,y) = x + y$ becomes.

15 Iterated integrals, Double integrals - 10.9

1 [T, §15.1, #9] Evaluate

$$\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx.$$

2 [T, §15.1, #15] Evaluate the double integral over the given region R

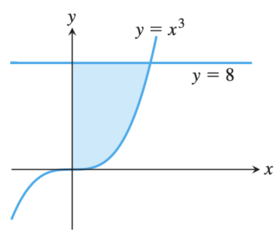
$$\iint_R (6y^2 - 2x)\,dA, \quad R: \quad 0 \le x \le 1, \quad 0 \le y \le 2$$

3 [T, §15.1, #33] Use Fubini's Theorem to evaluate

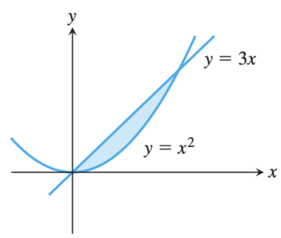
$$\int_0^2\int_0^1 \frac{x}{1+xy}\,dx\,dy.$$

4 Write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections :

a) [T, §15.2, #9]



b) [T, §15.2, #11]



c) [T, §15.2, #15] Bounded by $y = e^{-x}, y = 1$, and $x = \ln 3$.

5 [T, §15.2, #25] Evaluate the integral of $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x, y = 2x, x = 1$, and $x = 2$.

6 [T, §15.2, #27] Evaluate the integral of $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$.

7 [T, §15.2, #29] Sketch the region and evaluate

$$\int_{-2}^0\int_v^{-v} 2dpdv \quad \text{(the } pv\text{-plane)} .$$

16 More on double integrals - 10.11

1 [T, §15.2, #57] Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy-plane.

2 [T, §15.2, #59] Find the volume of the solid whose base is the region in the xy plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

3 [T, §15.2, #78] Evaluate the integral

$$\int_0^2 \left(\tan^{-1} \pi x - \tan^{-1} x\right) \, dx$$

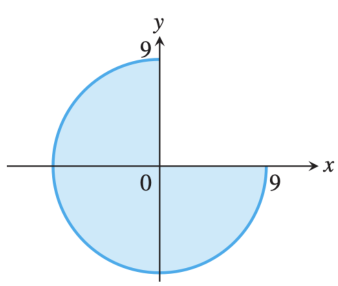
(Hint: Write the integrand as an integral and then use Fubini's theorem.)

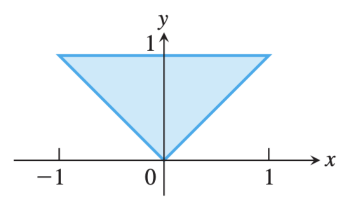
4 Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral :

a) [T, §15.3, #1] The coordinate axes and the line $x + y = 2$.

b) [T, §15.3, #9] The lines $y = x, y = x/3$ and $y = 2$.

5 [T, §15.4, #1, #3] Describe the given region in polar coordinates :





6 Change the Cartesian integral into an equivalent polar integral, then evaluate the polar integral :

a) [T, §15.4, #9]

$$\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy \, dx.$$

b) [T, §15.4, #19]

$$\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} \, dx \, dy.$$

7 [T, §15.4, #27] Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.

17 Change of variables, Jacobians - 10.16

1 [T, §15.8, #3]

a) Solve the system

$$u = 3x + 2y, \quad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b) Find the image under the transformation $u = 3x+2y$, $v = x+4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

c) [T, §15.8, #7] Use the transformation to evaluate the integral

$$\iint_R \left(3x^2 + 14xy + 8y^2\right) dxdy$$

for the region R in the first quadrant bounded by the lines

$$y = -(3/2)x + 1, \quad y = -(3/2)x + 3, \quad y = -(1/4)x, \quad y = -(1/4)x + 1$$

2 [T, §15.8, #11] Polar moment of inertia of an elliptical plate : A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1, a > 0, b > 0$, in the xy-plane. Find the second moment of inertia of the plate about the origin. (Hint: Use the transformation $x = ar\cos\theta, y = br\sin\theta$. The formula is $I = \iint x^2 + y^2\, dA$.)

3 Find the image of the set

$$S = \text{triangular region with vertices } (0,0),\ (1,1),\ (0,1) \text{ in the } uv\text{-plane}$$

under the transformation $x = u^2, y = v$.

4 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\int\int_R y\, dA$ where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$. (Assume $u, v \geq 0$ so we have a one-to-one mapping.)

18 Triple Integrals - 10.18

1 [T, §15.5, #3] Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.

2 [T, §15.5, #23] Find the volume of the region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, x = 1, y = -1, y = 1$.

3 [T, §15.5, Example 2] Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dydzdx$.

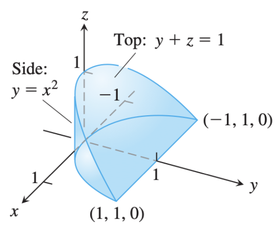
4 [T, §15.5, #5] Volume enclosed by paraboloids : Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

19 More on triple integrals - 10.23

1 [T, §15.5, #21] Rewrite the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dzdydx$$

as an equivalent iterated integral in the order a. $dydzdx$ b. $dydxdz$ c. $dxdydz$ d. $dxdzdy$ e. $dzdxdy$.



2 [T, §15.8, #20] Let D be the region in xyz-space defined by the inequalities

$$1 \le x \le 2, \quad 0 \le xy \le 2, \quad 0 \le z \le 1.$$

Evaluate

$$\iiint_D \left(x^2y + 3xyz\right) dxdydz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

3 [T, §15.7, #13] Give the limits of integration for evaluating the integral

$$\iiint f(r, \theta, z)\, dz\, rdr\, d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

4 [T, §15.7, Example 2] Find the mass of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

5 Find $\int\int\int_E zdV$ where E is the region bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = 4$.

6 Find the volume of the solid enclosed by the cylinder $x^2+y^2 = 9$ and the planes $y+z = 5$ and $z = 1$.

7 [T, §15.7, #33] Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \ge 0$.

20 Applications of triple integrals, Vector fields - 10.25

1 [T, §15.7, #67] A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r, r \geq 0$, and on the sides by the cylinder $r = 1$. Find the z-coordinate center of mass.

2 Find the volume of the solid that lies above the cone $\varphi = \pi/3$ and below the sphere $\rho = 4\cos\varphi$.

3 [T, §15.7, #69] Suppose the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\varphi = \pi/3$ has constant density 1. Find the z-coordinate center of mass of the solid by computation. Find the x, y-coordinate center of mass by symmetry.

4 Find the mass and set up the x-coordinate of the center of mass of the solid E with constant density function $\rho = 2$, where E lies under the plane $1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.

5 Which of the following vector fields are conservative :

- [T, §16.3, #1] $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$;
- [T, §16.3, #3] $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$;
- [T, §16.3, #5] $\mathbf{F} = (z + y)\mathbf{i} + z\mathbf{j} + (y + x)\mathbf{k}$.

21 Vector fields, Line integrals - 10.30

1 [T, §16.3, #9] Verify the following vector fields F are conservative and find the corresponding f such that $F = \nabla f$:

- $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$.

2 [T, §16.1, #9] Evaluate $\int_C (x+y)ds$ where C is the straight-line segment $x = t, y = (1-t), z = 0$, from $(0,1,0)$ to $(1,0,0)$.

3 [T, §16.1, #15] Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path C from $(0,0,0)$ to $(1,1,1)$ given by $C = C_1 \cup C_2$, where

$$C_1 : \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$$
$$C_2 : \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$$

4 Find the line integrals of $\mathbf{F}$:

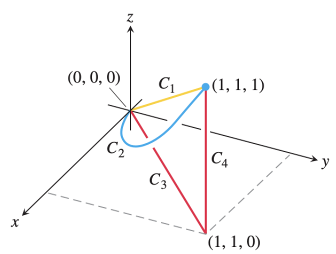
- [T, §16.2, #7] $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$,
- [T, §16.2, #10] $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

from $(0,0,0)$ to $(1,1,1)$ over each of the following paths in the accompanying figure :

a) the straight-line path $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$;

b) the curved path $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}, \quad 0 \le t \le 1$;

c) the path $C_3 \cup C_4$ consisting of the line segment from $(0,0,0)$ to $(1,1,0)$ followed by the segment from $(1,1,0)$ to $(1,1,1)$.



5 A counterexample to the criterion of conservative vector fields without the simply connected assumptions : verify

$$F(x,y) = \left(-\frac{y}{(x^2+y^2)}, \frac{x}{(x^2+y^2)} \right), \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

satisfies the condition $P_y = Q_x$ criterion. In fact, we will see later using line integrals and Green's theorem that this is not a conservative vector field.

22 Fundamental Theorem of line integrals, Green's theorem - 11.1

1 [T, §16.3, #19] Evaluate

$$\int_{(1,1,1)}^{(1,2,3)} 3x^2 dx + \frac{z^2}{y} dy + 2z \ln y dz$$

using the fundamental theorem of line integrals.

2 [T, §16.3, #21] Evaluate

$$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) dy - \frac{y}{z^2} dz$$

using the fundamental theorem of line integrals.

3 [T, §16.4, #1] Verify Green's theorem for the field

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$$

with the domains of integration to be the disk $R : x^2 + y^2 \leq a^2$ and its bounding circle $C :\ \mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.

4 [T, §16.4, #21] Evaluate

$$\oint_C \left(y^2 dx + x^2 dy\right)$$

using Green's theorem, where C is the triangle bounded by $x = 0, x + y = 1, y = 0$.

5 [T, §16.4, #23] Evaluate

$$\oint_C (6y + x)dx + (y + 2x)dy$$

using Green's theorem, where C is the circle $(x - 2)^2 + (y - 3)^2 = 4$.

23 Curl and divergence, Parametric surfaces - 11.13

1 Find the curl of $\overrightarrow{F}(x,y) = \sin(xy)\hat{i} + \cos(xy)\hat{j}$.

2 Find the curl of $\overrightarrow{F}(x,y,z) = xy\ \hat{i} + x^2z\ \hat{j} - (y+z^3)\ \hat{k}$.

3 [§16.5, #29] Verify the identity :

$$\text{curl (curl } F) = \text{grad}\,(\text{div } F) - \nabla^2 F.$$

4 Find a parametrization of the surfaces :

a) [T, §16.5, #1] The paraboloid : $z = x^2 + y^2, \quad z \leq 4$;

b) [T, §16.5, #5] Spherical cap : The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2 + y^2}$;

c) [T, §16.5, #13] Tilted plane inside cylinder : The portion of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 9$;

24 Surface area, Surface integrals - 11.15

1 Set up the integral for finding the area of the part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, $z = 1$.

2 Set up the integral for the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.

3 Evaluate the surface integral $\int\int_S y \, dS$ where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$.

4 Evaluate the flux of $\overrightarrow{F}(x, y, z) = xze^y \, \hat{\mathbf{i}} - xze^y \, \hat{\mathbf{j}} + z \, \hat{\mathbf{k}}$ across the surface S consisting of the part of the plane $x + y + z = 1$ in the first octant and with downward orientation (meaning $\hat{\mathbf{n}}$ has negative z component).

5 Find the flux of $\overrightarrow{F} = y \, \hat{\mathbf{i}} + (z - y) \, \hat{\mathbf{j}} + x \, \hat{\mathbf{k}}$ across the surface S which is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Take S to have outward orientation.

25 Stokes Theorem - 11.27

1 Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ above the plane $z = 0$, with upward normal. Let $\vec{F} = \langle y - z, -(x + z), x + y \rangle$. Compute $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$.

2 Find $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$ where $\vec{F} = (xyz)\hat{\mathbf{i}} + (xy)\hat{\mathbf{j}} + (x^2yz)\,\hat{\mathbf{k}}$ and S consists of the top and 4 sides (no bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ oriented outward.

3 Let C be the triangle in $\mathbb{R}^3$ with vertices $(1, 0, 0), (0, 2, 0)$ and $(0, 0, 1)$. Compute

$$\int_C \left(x^2 + y\right) dx + yzdy + \left(x - z^2\right) dz.$$

4 Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = x^2z\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

26 Divergence Theorem - 11.29

1 Use the Divergence theorem to find the flux of $\overrightarrow{F} = (e^z + y^2 x)\ \hat{\mathbf{i}} + (\cos x + x^2 z)\ \hat{\mathbf{k}}$ through the surface S bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 1$.

2 Use the Divergence theorem to calculate the flux of $\overrightarrow{F} = |\overrightarrow{r}|\overrightarrow{r}$ through the surface S given by the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leq 1$ in the xy-plane. (Here $\overrightarrow{r}$ denotes the radial vector $\langle x, y, z \rangle$).

3 Use the divergence theorem to evaluate the flux integral $\int\int_S \overrightarrow{F} \cdot d\overrightarrow{S}$ where $\overrightarrow{F} = \langle yz, x^2 + y, z^2 \rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$. (Note S is not closed so you have to make it closed and subtract out the flux integral over the surface you added in.)

4 Use the divergence theorem to find the flux of $\overrightarrow{F} = y\ \hat{\mathbf{i}} + (z - y)\ \hat{\mathbf{j}} + x\ \hat{\mathbf{k}}$ across the surface S which is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Take S to have outward orientation.

27 Solutions to selected triple integral problems

1 [T, §15.5, #3] Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.

Solution :

$$\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dzdydx, \quad \int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dzdxdy,$$
$$\int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dydzdx, \quad \int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dydxdz,$$
$$\int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dxdzdy, \quad \int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dxdydz.$$

The value of all six integrals is 1 .

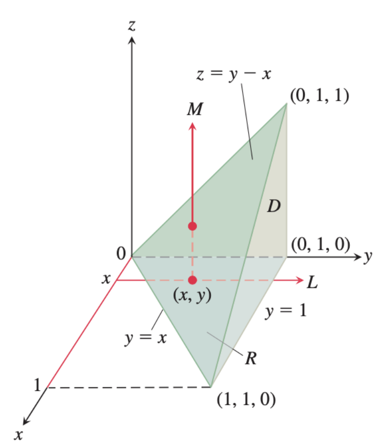
2 [T, §15.5, #23] Find the volume of the region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, x = 1, y = -1, y = 1$.

Solution : We compute

$$\int_0^1 \int_{-1}^1 \int_0^{y^2} 1 \, dz \, dy \, dx = \int_{-1}^1 y^2 \, dy = \frac{2}{3}.$$

3 [T, §15.5, Example 2] Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dydzdx$.

Solution : We sketch D along with its "shadow" R in the xz-plane. The upper (right-hand) bounding surface of D lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of R is the line $z = 1 - x$. The lower boundary is the line $z = 0$.



First we find the y-limits of integration. The line through a typical point (x, z) in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

Next we find the z-limits of integration. The line L through (x, z) parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. The integral is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) dydzdx.$$

4 [T, §15.5, #5] Volume enclosed by paraboloids : Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

Solution :

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} 1dzdxdy, \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{8-x^2-y^2} 1dzdxdy$$

$$\int_{-2}^{2} \int_{4}^{8-y^2} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1dxdzdy + \int_{-2}^{2} \int_{y^2}^{4} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1dxdzdy$$

$$\int_{4}^{8} \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1dxdydz + \int_{0}^{4} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1dxdydz$$

$$\int_{-2}^{2} \int_{4}^{8-x^2} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1dydzdx + \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1dydzdx$$

$$\int_{4}^{8} \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1dydxdz + \int_{0}^{4} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1dydxdz$$

The value of all six integrals is 16π.

5 [T, §15.7, Example 2] Find the mass of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution : We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$. Its base R is the disk $0 \le r \le 2$ in the xy-plane.

The value of M is

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} dzrdrd\theta = \int_0^{2\pi} \int_0^2 [z]_0^{r^2} rdrd\theta \\ &= \int_0^{2\pi} \int_0^2 r^3 drd\theta = \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4d\theta = 8\pi. \end{aligned}$$

6 [T, §15.7, #13] Give the limits of integration for evaluating the integral

$$\iiint f(r,\theta,z)\,dz\,rdr\,d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

Solution :

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r,\theta,z)dzrdrd\theta$$

7 Find $\int\int\int_E z\,dV$ where E is the region bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = 4$.

Solution : We compute

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^1\int_{4r^2}^4 z\,dzr\,dr\,d\theta = \frac{16\pi}{3}.$$

8 Find the volume of the solid enclosed by the cylinder $x^2+y^2 = 9$ and the planes $y+z = 5$ and $z = 1$.

Solution : We compute

$$\int_0^{2\pi}\int_0^3\int_1^{5-r\sin\theta} 1\,dzr\,dr\,d\theta == \int_0^{2\pi}\int_0^3 (4 - r\sin\theta)r\,dr\,d\theta = 36\pi$$

9 [T, §15.7, #67] A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r, r \geq 0$, and on the sides by the cylinder $r = 1$. Find the z-coordinate center of mass.

Solution : First, we compute the mass :

$$M = \int_0^{2\pi}\int_0^1\int_0^r dz\,r\,dr\,d\theta = 2\pi\int_0^1 r^2\,dr = \frac{2\pi}{3}$$

and then we write

$$\bar{z} = \frac{1}{M}\iiint z\,dV = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^r z\,dz\,r\,dr\,d\theta = \frac{1}{M}\pi\int_0^1 r^3\,dr = \frac{3}{8}.$$

10 Find the volume of the solid that lies above the cone $\varphi = \pi/3$ and below the sphere $\rho = 4\cos\varphi$.

Solution : We compute

$$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{64}{3}\cos^3\varphi\sin\varphi\,d\varphi\,d\theta$$

$$= \frac{128\pi}{3}\int_0^{\pi/3}\cos^3\varphi\sin\varphi\,d\varphi = -\frac{32\pi}{3}\cos^4\varphi|_0^{\pi/3} = -\frac{32\pi}{3}(1/16 - 1) = 10\pi.$$

11 [T, §15.7, #69] Suppose the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\varphi = \pi/3$ has constant density 1. Find the z-coordinate center of mass of the solid by computation. Find the x, y-coordinate center of mass by symmetry.

Solution : We compute the mass

$$M = \int_0^{2\pi} \int_{\pi/3}^{\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{16\pi}{3} \int_{\pi/3}^{\pi} \sin\varphi \, d\varphi = 8\pi.$$

Then we compute

$$\bar{z} = \frac{1}{M} \int_0^{2\pi} \int_{\pi/3}^{\pi} \int_0^2 \rho \cos\varphi \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{8\pi}{M} \int_{\pi/3}^{\pi} \cos\varphi \sin\varphi \, d\varphi = \frac{1}{4} \int_{2\pi/3}^{2\pi} \sin\varphi \, d\varphi = \frac{3}{8}.$$

By symmetry, $\bar{x} = \bar{y} = 0$.

12 Find the mass and set up the x-coordinate of the center of mass of the solid E with constant density function $\rho = 2$, where E lies under the plane $1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.

Solution : The mass is given by

$$m = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2 \, dzdydx = 2 \int_0^1 (1+x)\sqrt{x} + \frac{1}{2} x \, dx = 2 \left(\frac{2}{3} x^{3/2} + \frac{2}{5} x^{5/2} + \frac{1}{4} x^2 \right) |_0^1 = \frac{79}{30}.$$

The center of mass is given by

$$\overline{x} = \frac{1}{m} \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x \, dzdydx.$$

28 Acknowledgements

In this note, the problems are mainly cited from our textbook and the book *Thomas' Calculus: Multivariable, 14th edition*. For these kinds of problems, we cite the question number so that you can locate it if you want. Note that, Thomas' book contains the solutions to odd numbered exercises.