NOTES FOR MATH 222B: PARTIAL DIFFERENTIAL EQUATIONS, SPRING 2021

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This note is a transcription for Math 222B taught by Professor Maciej Zworski in Spring 2021 at UC Berkeley. I sincerely appreciate Professor Zworski for his excellent lectures and his generous help during the office hours. However, I realized that the material was so important and finally decided to type down the notes one year later.

As a continuation of Math 222A, this class reviews the theory of distributions and covers the rigorous mathematical theory of calculus of variations and microlocal analysis. The primary reference is [2] and [4].

Contents

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1. Review of Distribution Theory

1.1. Distributions and Sobolev Spaces.

Definition 1.1. *For an open set $U \subset \mathbb{R}^n$, $u \in \mathcal{D}'(U)$ means that $u : C_c^\infty(U) \to \mathbb{C}$, satisfying that*

$$\forall K \Subset U, \exists C, N, \forall \varphi \in C_c^\infty(K),\ |u(\varphi)| \leq C \sup_{|\alpha| \leq N, x \in K} |\partial^\alpha \varphi(x)|.$$

In other words, u is a functional on compactly supported smooth functions with this type of estimate, and the constant depends on the compact set.

Definition 1.2 (Distributions of order k)**.** *We say $u \in \mathcal{D}'^{(k)}(U)$ if $\forall K \Subset U$, there exists C such that for all $\varphi \in C_c^\infty(K)$,*

$$|u(\varphi)| \leq C \sup_{|\alpha| \leq k, x \in K} |\partial^\alpha \varphi(x)|.$$

Example 1.3. *Take $U = (0,1) \subset \mathbb{R}$,*

$$u = \sum_n \delta_{\frac{1}{n}},$$

where $\delta_{\frac{1}{n}}(\varphi) = \varphi(\frac{1}{n})$. Note that the sum does not converge for a smooth function compactly supported in $\mathbb{R}$, but it converges for a smooth function compactly supported in $(0,1)$.

Example 1.4. *If $u \in L^1_{loc}(U)$, we define*

$$u(\varphi) = \int u\varphi.$$

In this case, a distribution is a function.

The magic of distributions is that we can differentiate as many times as we want. The differentiation is defined by formal integration by parts:

Definition 1.5.

$$\partial^\alpha u(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi).$$

This is the basic of Sobolev spaces:

Definition 1.6.

$$W^{k,p}(U) = \{u \in L^1_{loc}(U) : \partial^\alpha u \in L^p(U), \forall |\alpha| \leq k\},$$

where $k \in \mathbb{N}_0, 1 \leq p \leq \infty$, and the derivative $\partial^\alpha u$ is taken in the sense of distributions. And

$$H^k(U) = W^{k,2}(U)$$

which is a Hilbert space with $\langle u, v\rangle_{H^k} = \sum_{|\alpha| \leq k} \int_U \partial^\alpha u \overline{\partial^\alpha v}$.

Definition 1.7.

$$W_0^{k,p}(U) := \overline{C_c^\infty(U)}^{W^{k,p}}, \quad H_0^1(U) := \overline{C_0^\infty(U)}^{H^1}.$$

Theorem 1.8 (Approximation). *(1) For $U \Subset \mathbb{R}^n$,*

$$\overline{C^\infty(U) \cap W^{k,p}(U)}^{W^{k,p}} = W^{k,p}(U).$$

(2) For $U \Subset \mathbb{R}^n$, ∂U is C^1,

$$\overline{C^\infty(\overline{U}) \cap W^{k,p}(U)}^{W^{k,p}} = W^{k,p}(U).$$

Theorem 1.9 (Extension). *For $U \Subset \mathbb{R}^n$, ∂U is C^1, for any open set V, such that $\overline{U} \Subset V \Subset \mathbb{R}^n$, there exists $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ bounded and linear such that $Eu|_U = u$ and $\operatorname{supp} u \Subset V$. Moreover, if we want to extend a function in $W^{k,p}$, we need ∂U is C^k.*

Theorem 1.10 (Traces). *For $U \Subset \mathbb{R}^n$, ∂U is C^1, there exists $T : W^{1,p}(U) \to L^p(\partial U)$ bounded and linear such that*

$$Tu = u|_{\partial U}, \text{ for } u \in C(\overline{U}).$$

Example 1.11 (Characterization of $H_0^1(U)$). *For $U \Subset \mathbb{R}^n$, ∂U is C^1,*

$$H_0^1(U) = \{u \in H^1 : Tu = 0 \text{ in } L^2(\partial U)\}.$$

Remark 1.12. The trace theorem is not optimal in the sense that we can do better for the image, i.e. we can find a function space between the image of T and $L^p(\partial U)$. Intuitively, restriction to the boundary loses some Sobolev regularities. We can get a feeling from Theorem 1.23 below. That is, we will show that for $U = \mathbb{R}^n_+$ with $\partial U = \mathbb{R}^{n-1}$, we have

$$T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}).$$

We will discuss this by using Fourier transform.

1.2. **Schwartz class and Tempered distributions.**

Definition 1.13 (Schwartz class). *Schwartz spaces is a class of functions*

$$\mathscr{S} = \{\varphi \in C^\infty(\mathbb{R}^n) : x^\alpha \partial^\beta \varphi \in L^\infty, \forall \alpha, \beta \in \mathbb{N}^n\}.$$

The condition for φ is equivalent to

$$|\partial^\alpha \varphi(x)| \leq C_{N,\alpha}(1 + |x|)^{-N}, \ \forall N.$$

The Fourier transform of Schwartz functions $\varphi \in \mathscr{S}$ is given by

$$\widehat{\varphi}(\xi) := \int \varphi(x) e^{-ix\cdot\xi}\, dx.$$

And sometimes we write $\widehat{\varphi} = \mathcal{F}\varphi$. Since $\mathcal{F}(\frac{1}{i}\partial_x \varphi) = \xi \mathcal{F}\varphi$, $\mathcal{F}(x\varphi) = -\frac{1}{i}\partial_\xi \mathcal{F}\varphi$, we have

$$\mathcal{F} : \mathscr{S} \to \mathscr{S}.$$

And its inverse is

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} R\mathcal{F}, \quad R\varphi(x) = \varphi(-x),$$

which gives the Fourier inversion formula

$$\varphi(x) = \frac{1}{(2\pi)^n} \int \widehat{\varphi}(\xi) e^{ix\cdot\xi}\, d\xi,$$

the cornerstone of all other things.

Definition 1.14 (Tempered distribution)**.** *We say $u \in \mathscr{S}'$ if $u : \mathscr{S} \to \mathbb{C}$ and there exists C, N, such that*

$$|u(\varphi)| \le C \sup_{|\alpha|,|\beta| \le N} |x^\alpha \partial^\beta \varphi|.$$

Equivalently, we can replace the RHS by $(1+|x|)^N \sup_{|\beta|\le N} |\partial^\beta \varphi|$.

Of course, $\mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ since $\mathscr{S}(\mathbb{R}^n) \supset C_c^\infty(\mathbb{R}^n)$. Define $\mathcal{F} : \mathscr{S}' \to \mathscr{S}'$ by $\widehat{u}(\varphi) = u(\widehat{\varphi})$.

Proposition 1.15.

$$\langle \widehat{u}, \widehat{\varphi} \rangle_{L^2} = (2\pi)^n \langle u, \varphi \rangle_{L^2}, \quad u, \varphi \in \mathscr{S}.$$
$$\|\widehat{u}\|_{L^2} = (2\pi)^{\frac{n}{2}} \|u\|_{L^2}, \quad u \in \mathscr{S}.$$

Now if $u_n \xrightarrow{L^2} u$, we have $u_n \xrightarrow{\mathscr{S}'} u$ since

$$|(u_n - u)(\varphi)| = |\int (u_n-u)\varphi| \le \|u_n - u\|_{L^2} \|\varphi\|_{L^2} \le \|u_n - u\|_{L^2} \|(1+|x|)^{-N}\|_{L^2} \sup\left((1+|x|)^N \varphi\right).$$

And hence $\widehat{u_n} \xrightarrow{\mathscr{S}'} \widehat{u}$. This implies that $\mathcal{F} : L^2 \to L^2$ and

$$\langle \widehat{u}, \widehat{v} \rangle = (2\pi)^n \langle u, v \rangle_{L^2}.$$

Example 1.16.

$$\widehat{\delta_0}(\varphi) = \delta_0(\widehat{\varphi}) = \widehat{\varphi}(0) = \int 1\varphi = 1(\varphi),$$

where $1 = \widehat{\delta_0}$ is a tempered distribution.

Example 1.17. *Consider $\mathbb{R}^2$, $u(x) = \frac{1}{|x|}$. Then u is a tempered distribution since $u \in L^1_{loc}$ and $(1+|x|)^{-2} u \in L^1$, which implies*

$$|u(\varphi)| = |\int (1+|x|)^{-2} u \ (1+|x|)^2 \varphi| \le C \sup (1+|x|)^2 |\varphi|.$$

Now we would like to compute the Fourier transform of u. We will use the trick that Fourier transform is continuous. In order to do this, we try to find $u_\varepsilon \in L^1$ such that $u_\varepsilon \xrightarrow{\mathscr{S}'} u$. Take $u_\varepsilon(x) = \frac{1}{|x|} e^{-\frac{1}{2}\varepsilon |x|^2} \in L^1$, $\varepsilon > 0$. Note that

$$\begin{aligned}
\widehat{u_\varepsilon}(\xi) &= \int_{\mathbb{R}^2} \frac{1}{|x|} e^{-\frac{1}{2}\varepsilon|x|^2 - ix\cdot\xi}\, dx = \int_0^{2\pi} \int_0^\infty \frac{1}{r} e^{-\frac{1}{2}\varepsilon r^2 - ir(\cos\theta \xi_1 + \sin\theta \xi_2)} r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^\infty e^{-\frac{\varepsilon}{2}(r + i\frac{\cos\theta\xi_1 + \sin\theta\xi_2}{\varepsilon})^2 - \frac{1}{2\varepsilon}(\cos\theta\xi_1 + \sin\theta\xi_2)^2}\, dr\, d\theta \\
&= \int_0^{2\pi} e^{-\frac{1}{2\varepsilon}(\cos\theta\xi_1 + \sin\theta\xi_2)^2} \int_0^\infty e^{-\frac{\varepsilon}{2}r^2}\, dr\, d\theta = \frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} \int_0^{2\pi} e^{-\frac{|\xi|^2}{2\varepsilon}(\cos\theta\cos\varphi + \sin\theta\sin\varphi)^2}\, d\theta \\
&= \frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} \int_0^{2\pi} e^{-\frac{1}{2\varepsilon}|\xi|^2 \cos^2\theta}\, d\theta,
\end{aligned}$$

where we complete the square in the third equality and we deform the contour in the fourth euqality. We cannot calculate the last integral, but we do know the asymptotic behavior as

$\varepsilon \to 0$, in the following lectures. We will learn how to do asymptotics even though we cannot evaluate this integral:

$$\int_0^{2\pi} e^{-\frac{1}{2\varepsilon}|\xi|^2\cos^2\theta}\, d\theta = 2\sqrt{2\pi}|\xi|^{-1}\sqrt{\varepsilon}(1+O(\varepsilon)).$$

And we get

$$\widehat{u}(\xi) = \frac{2\pi}{|\xi|}.$$

Now we need another method to find $\widehat{u}(\xi)$. Note that u is homogeneous, which is a very fortune property. We say a function v has homogeneity of degree a if $v(tx) = t^a v(x)$ for $t > 0$. By making a change of variable,

$$t^n \int v(x)\varphi(x)\, dx = \int v(tx)\varphi(x)\, dx = \int v(y)\varphi(\frac{y}{t})t^{-n}\, dy.$$

For distributions v, we say $v \in \mathscr{S}'$ is homogeneous of degree a if $\forall \varphi \in \mathscr{S}$,

$$u\left(\varphi(\frac{\cdot}{t})t^{-n}\right) = t^a u(\varphi),$$

for $t > 0$. However, in practice, we can just manipulate things as if they were functions. If $v \in \mathscr{S}'$, homogeneous of degree a, then

$$\widehat{v}(t\xi) = \int v(x)e^{-ix\cdot t\xi}\, dx = \int t^{-a}v(tx)e^{-itx\cdot\xi}\, t^{-n} d(tx) = t^{-n-a}\widehat{v}(\xi).$$

This implies $\widehat{v}$ is homogeneous of degree $-n-a$.

Now we follow the development in [7, Chapter 1]*: We will call a distribution v of class a if it is homogeneous of degree a and C^∞ on $\mathbb{R}^n \setminus \{0\}$. Then we have a more powerful result (c.f.* [7, Theorem 1, Chapter 1]*):*

Theorem 1.18. *v is of class a if and only if $\widehat{v}$ is of class $-n-a$.*

Proof. Let v_0 denote the C^∞ function on $\mathbb{R}^n \setminus \{0\}$ that agrees with v there. Choose $\psi \in C_c^\infty$ such that $\psi \equiv 1$ in a neighborhood of 0. Then $\widehat{v}(\xi) = \widehat{\psi v}(\xi) + \widehat{(1-\psi)v}(\xi)$, where $\widehat{\psi v}(\xi)$ is C^∞ since ψv has compact support. And using homogeneity of v, we know that $\Delta^k((1-\psi)v) \in L^1$ for $k > n + a$, which implies

$$\left(\Delta^k((1-\psi)v)\right)^\wedge = (-|\xi|^2)^k((1-\psi)v)^\wedge(\xi)$$

is continuous. Thus, $((1-\psi)v)^\wedge(\xi)$ is continuous on $\mathbb{R}^n \setminus \{0\}$. Hence, $\widehat{v}(\xi)$ is continuous on $\mathbb{R}^n \setminus \{0\}$. □

In consequence, $\widehat{\frac{1}{|x|}}$ is homogeneous of degree -1, that is,

$$\widehat{u}(r\theta) = \frac{a(\theta)}{r},$$

where $\theta \in S^1$, $r > 0$. Moreover, u is rotational symmetric, which implies

$$\widehat{u}(\xi) = \int u(x)e^{-ix\cdot\xi}\, dx = \int u(R_\theta x)e^{-ix\cdot\xi}\, dx = \int u(y)e^{-ix\cdot R_{-\theta}\xi}\, dy = \widehat{u}(R_{-\theta}\xi).$$

Then $\widehat{u}$ is invariant under rotation and homogeneity of degree -1, which implies $a(\theta)$ is constant, and hence

$$\widehat{u}(\xi) = \frac{c}{|\xi|}, \quad \text{for } \xi \neq 0.$$

Since the only function supported only at 0 is $\delta_0^{(\alpha)}$,

$$\widehat{\frac{1}{|x|}} = \frac{c}{|\xi|} + \sum_{|\alpha| \leq N} c_\alpha \delta_0^{(\alpha)}.$$

Note that $\delta_0(x)$ is homogeneous of degree -2 in $\mathbb{R}^2$, which can be checked formally:

$$\int_{\mathbb{R}^n} \delta_0(tx) f(x)\, dx = t^{-n} \int_{\mathbb{R}^n} \delta_0(y) f(\frac{y}{t})\, dy = \delta_0(f(\frac{\cdot}{t})t^{-n}) = f(0)t^{-n} = t^{-n}\delta_0(f).$$

Moreover, $\delta_0^{(\alpha)}$ in $\mathbb{R}^2$ is homogeneous of degree $-2-|\alpha|$. Since $-2-|\alpha| < -1$ for all α, that weird term cancelled and finally

$$\widehat{\frac{1}{|\xi|}} = \frac{c}{|\xi|}.$$

Now we calculate the constant c. Note that $\langle \widehat{u}, \widehat{\varphi} \rangle = (2\pi)^2 \langle u, \varphi \rangle$ is also true for $u \in \mathscr{S}', \varphi \in \mathscr{S}$. We choose φ to be the Gaussian $\varphi(x) = e^{-\frac{|x|^2}{2}}$, and LHS should be

$$\int_{\mathbb{R}^2} \frac{1}{|x|} \varphi(x)\, dx = \int_0^{2\pi} \int_0^\infty \varphi(r)\, dr\, d\theta = 2\pi \int_0^\infty e^{-\frac{r^2}{2}}\, dr = \frac{1}{2}(2\pi)^{\frac{3}{2}}.$$

Since $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^2} e^{-|x|^2/2 - ix\cdot\xi}\, dx = \int_{\mathbb{R}^2} e^{-\frac{1}{2}|x+i\xi|^2 - \frac{1}{2}|\xi|^2}\, dx = 2\pi e^{-\frac{1}{2}|\xi|^2}$,

$$c \int \frac{1}{|\xi|} 2\pi e^{-\frac{1}{2}|\xi|^2}\, d\xi = 2\pi c \frac{1}{2}(2\pi)^{\frac{3}{2}} = (2\pi)^2 \frac{1}{2}(2\pi)^{\frac{3}{2}},$$

which implies $c = 2\pi$.

1.3. Characterization of $H^k(\mathbb{R}^n)$ using the Fourier transform.

Definition 1.19 (Sobolev spaces)**.**

$$W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) : \partial^\alpha u \in L^2, |\alpha| \leq k\}.$$

Note that for $k \in \mathbb{N}$, those in H^k are all functions since they are in L^2 (or more precisely, they are identified with L^2 functions). Here we write $u \in \mathcal{D}'(\mathbb{R}^n)$ is to specify the derivative in the definition is in the sense of distribution.

Proposition 1.20.

$$H^k(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : (1+|\xi|^2)^{\frac{k}{2}} \widehat{u} \in L^2\}.$$

Proof. Suppose $\partial^\alpha u \in L^2, |\alpha| \leq k$, then $\widehat{\partial^\alpha u} = i^{|\alpha|}\xi^\alpha u \in L^2$, $\forall |\alpha| \leq k$. Since $(1+|\xi|^2)^k \leq C_k \sum_{l \leq 2k} |\xi|^l \leq C_{n,k} \sup_{|\alpha| \leq k} |\xi^\alpha|^2$, we have $(1+|\xi|^2)^{\frac{k}{2}} \widehat{u} \in L^2$.

Now suppose $(1+|\xi|^2)^{\frac{k}{2}} \widehat{u} \in L^2$, and by the same type of thinking, $|\xi^\alpha| \leq |\xi|^{|\alpha|} \leq (1+|\xi|)^k \leq C_k(1+|\xi|^2)^{\frac{k}{2}}$, for $|\alpha| \leq k$, we know that $\partial^\alpha u = \mathcal{F}^{-1}\left(i^{|\alpha|}\xi^\alpha \widehat{u}(\xi)\right) \in L^2$. ☐

Notation 1.21 (Japanese bracket)**.**

$$\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}.$$

The definition above does not require $k \in \mathbb{N}$, and it works for any real number.

Definition 1.22 (Sobolev spaces)**.**

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle \xi \rangle^s \widehat{u} \in L^2\}, \quad s \in \mathbb{R}.$$

Theorem 1.23. *Suppose $u \in H^s(\mathbb{R}^n)$ and $s > \frac{1}{2}$, and we define $Tu(y) = u(0,y)$ for $u \in \mathscr{S}$, $y \in \mathbb{R}^{n-1}$ then $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. In particular, for $u \in \mathscr{S}$, $v(y) = u(0,y) \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. Take $u \in \mathscr{S}$, we need to prove

$$\|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

Since $\widehat{v}(\eta) = \int_{\mathbb{R}^{n-1}} u(0,y)e^{-iy\cdot\eta}\,dy = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{u}(\xi_1,\eta)\,d\xi_1$, where the last equality follows from $f(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\,d\xi$ for dimension 1. We compute

$$\begin{aligned}
\|v\|^2_{H^{s-\frac{1}{2}}} &= \int_{\mathbb{R}^{n-1}} \langle\eta\rangle^{2s-1}|\widehat{v}(\eta)|^2\,d\eta = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^{n-1}} \langle\eta\rangle^{2s-1}\left|\int_{\mathbb{R}} \widehat{u}(\xi_1,\eta)\,d\xi_1\right|^2 d\eta \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^{n-1}} \langle\eta\rangle^{2s-1}\left|\int_{\mathbb{R}} \widehat{u}(\xi_1,\eta)(1+|\xi_1|^2+|\eta|^2)^{\frac{s}{2}}(1+|\xi_1|^2+|\eta|^2)^{-\frac{s}{2}}\,d\xi_1\right|^2 d\eta \\
&\leq \frac{1}{(2\pi)^2}\int_{\mathbb{R}^{n-1}} \langle\eta\rangle^{2s-1}\int_{\mathbb{R}} \left(\widehat{u}(\xi_1,\eta)\right)^2(1+|\xi_1|^2+|\eta|^2)^s\,d\xi_1\int_{\mathbb{R}}(1+|\xi_1|^2+|\eta|^2)^{-s}\,d\xi_1\,d\eta,
\end{aligned}$$

where

$$\int \left(1+|\eta|^2+|\xi_1|^2\right)^{-s}\,d\xi_1 = \langle\eta\rangle^{-2s+1}\int\left(1+\left|\frac{\xi_1}{\langle\eta\rangle}\right|^2\right)^{-s} d\frac{\xi_1}{\langle\eta\rangle} = C_s\langle\eta\rangle^{-2s+1}.$$

Thus $\|v\|^2_{H^{s-\frac{1}{2}}} \leq \frac{1}{(2\pi)^2}C_s\int\langle\xi\rangle^{2s}\left(\widehat{u}(\xi_1,\eta)\right)^2\,d\xi_1\,d\eta = C_s\|u\|^2_{H^s}$. □

Remark 1.24. This theorem tells us restriction loses half regularity.

Theorem 1.25. *If $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ denotes the continuous functions that tends to 0 as $|x| \to \infty$.*

Proof. The steps are as follows:

(1) $\langle\xi\rangle^s\widehat{u} \in L^2$ with $s > \frac{n}{2} \Rightarrow \widehat{u} \in L^2$;
(2) $\widehat{u} \in L^1 \Rightarrow u \in L^\infty$;
(3) u is continuous;
(4) u tends to 0 at the infinity.

Step 1: We apply the oldest trick in the book - multiply and divide,

$$\int |\widehat{u}|\,d\xi = \int \langle\xi\rangle^{-s}\langle\xi\rangle^s|\widehat{u}|\,d\xi \leq C\|u\|_{H^s}.$$

Step 2: From the Fourier inversion formula, $u(x) = \frac{1}{(2\pi)^n}\widehat{u}(\xi)e^{ix\cdot\xi}\,d\xi$, we have $|u(x)| \leq C\|\widehat{u}\|_{L^1}$.

Step 3: Since $x \mapsto \widehat{u}(\xi)e^{ix\cdot\xi}$ is continuous, $x \mapsto u(x)$ is continuous by the Dominated Convergence Theorem.

Step 4: We apply the Riemann-Lebesgue Lemma. $\square$

Lemma 1.26 (Riemann-Lebesgue Lemma)**.** *If $\widehat{u} \in L^1$, then $u(x) \to 0$ as $|x| \to \infty$.*

Proof. Recall $\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense. (This is an important fact. Actually, $\mathbb{C}_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense. The proof is sketched as follows: One first truncate the L^1 function v to a certain ball of radius R, denoted by $v_R = v1_{B(0,R)}(x)$, then we get v_R converges to v in L^1 norm as $R \to \infty$ by the Dominated Convergence Theorem. And now we take the approximation of identity $\varphi \in C_c^\infty, \varphi \geq 0, \int \varphi = 1$ with $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$. Let $v_{R,\varepsilon} = v_R * \varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, then $v_{R,\varepsilon} \to v_R$ in L^1 as $\varepsilon \to 0$.) Then there exists $v \in \mathscr{S}$, $\|\widehat{v} - \widehat{u}\|_{L^1} < \varepsilon$. Take R such that $|v(x)| < \varepsilon$ for $|x| > R$. Then $|u(x)| \leq |u(x) - v(x)| + |v(x)| \leq C\varepsilon + \varepsilon$ for $|x| > R$. $\square$

Remark 1.27 (Sanity check). When $n = 1$, the above two theorems imply if $u \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, then $u \in C(\mathbb{R})$ and $u(0)$ is well-defined if $s > \frac{1}{2}$(there is no y when $n = 1$). These two results are consistent since you can only evaluate at one point if the function is continuous.

1.4. **GNS inequality and Morrey's inequality.**

Theorem 1.28 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $1 \leq p < n$, $p^* = \frac{np}{n-p}$, then there exists $C = C(n,p)$ such that*

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^p}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Furthermore, if $U \Subset \mathbb{R}^n$, ∂U is C^1, then there exists $C = C(n,p,U)$ such that

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Remark 1.29. We can get the value of p^* by scaling ("dimensional analysis"). Take $u_\lambda(x) = u(\lambda x)$, then $\|u_\lambda\|_{L^{p^*}} \leq C\|\nabla u_\lambda\|_{L^p}$. Since $\|u_\lambda\|_{L^{p^*}} = \lambda^{-n/p^*}\|u\|_{L^{p^*}}$ and $\|\nabla u_\lambda\|_{L^p} = \lambda^{1-n/p}\|\nabla u\|_{L^p}$, we have $1 - n/p = -n/p^*$.

And this is false for $p = n > 1, p^* = \infty$ where the counterexample is $u(x) = \log\log(1 + |x|^{-1})\chi(x)$, where $\chi \in C_0^\infty$ which is 1 near 0. This follows from the result $\int_1^\infty \frac{dr}{r\log^\alpha r} < \infty$ for $\alpha > 1$. It's OK for $p = n = 1, p^* = \infty$ by the fundamental theorem of calculus.

On the other hand, Morrey's inequality treats the opposite case when $p > n$. We notice that when $p < n$, we miss L^∞. When $p > n$, we get more, we get Holder's continuity.

Theorem 1.30 (Morrey's inequality)**.** *Let $n < p \leq \infty$. Then there exists $C = C(p,n)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\left(\|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}\right)$$

for all $u \in C^1(\mathbb{R}^n)$, $\gamma = 1 - \frac{n}{p}$. Here $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} := \sup|u| + \sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\gamma}$. More specifically, suppose $u \in W^{1,p}(\mathbb{R}^n)$, then there exists $u^* \in C^{0,\gamma}(\mathbb{R}^n)$, $\gamma = 1 - \frac{n}{p}$ such that

$$u = u^* \text{ a.e. and } \|u\|_{C^\gamma(\mathbb{R}^n)} \leq C\left(\|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}\right).$$

Sometimes we forget to mention u^*.

This inequality comes from Berkeley. Although we will not use Morrey's inequality in this course, we will present a different proof using the Fourier transform while the proof in [2] uses the real variable method. We will defer the proof to next section.

Theorem 1.31 (General formulation)**.** *Let $U \Subset \mathbb{R}^n, \partial U$ is C^1. For $u \in W^{k,p}(\mathbb{R}^n)$, we have the following statements:*

(1) $k < \frac{n}{p} \Rightarrow u \in L^q(U), \frac{1}{q} \geq \frac{1}{p} - \frac{k}{n}$ and $\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}$;

(2) $k > \frac{n}{p} \Rightarrow u \in C^{k-\frac{n}{p}-1,\gamma}(\overline{U})$, where $\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \notin \mathbb{N} \\ 1-\delta, & \forall \delta > 0 \text{ if } \frac{n}{p} \notin \mathbb{N} \end{cases}$.

Proof. The general formulation comes from two steps. The first step is that you start with u and you take an extension of u, and then you approximate your extension by a smooth function of compact support of the extension. You apply this theorem with some iteration and you will get this statement. □

Remark 1.32. Since U is bounded, the larger q we have, the L^q will be better, that is, $L^{q_1}(U) \subset L^{q_2}(U)$ for $q_1 > q_2$. Indeed, we only need to consider $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ in the first case.

If U is unbounded, we will get local results like $u \in L^q_{loc}(U)$ in the first case.

1.5. **A different approach to prove Morrey's inequality.** Before we present the proof, we make a quick a review for some basic inequalities at first.

Proposition 1.33. *(1) Holder inequality:*

$$|\int fg| \leq \|f\|_p\|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(2) Minkowski inequality:

$$\text{Version 1: } \|f+g\|_p \leq \|f\|_p + \|g\|_p,$$

$$\text{Version 2: } \|\int F(\cdot,t)\,dt\|_p \leq \int \|F(\cdot,t)\|_p\,dt.$$

(3) Young's inequality:

$$\|f * g\|_p \leq \|f\|_1\|g\|_p.$$

More generally, for $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, we have

$$\|f * g\|_r \leq \|f\|_p\|g\|_q.$$

Proof. We prove Young's inequality as follows.

$$\begin{aligned} &\int \left(\int |f(x-y)g(y)|\,dy\right)^p dx \\ \leq &\int \left(\int |f(x-y)|^{1-\frac{1}{p}}|f(x-y)|^{\frac{1}{p}}|g(y)|\,dy\right)^p dx \\ \leq &\int \|f\|_1^{p-1} \int |f(x-y)||g(y)|^p\,dy\,dx = \|f\|_1^p\|g\|_p^p. \end{aligned}$$

□

Now we need to introduce the Littlewood-Paley decomposition.

Lemma 1.34 (Dyadic partition of identity)**.** *There exists $\psi_0 \in C_c^\infty(\mathbb{R})$, $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$, such that*

$$\psi_0(|\xi|) + \sum_{j=0}^{\infty} \psi(2^{-j}|\xi|) = 1.$$

Proof. Choose $\varphi_0 \in C_c^\infty((-1,1))$ such that $0 \le \varphi_0 \le 1$ and $\varphi(\rho) = 1$ for $|\rho| \le \frac{1}{2}$. Then we define

$$\varphi_1(\rho) := \sum_{j \in \mathbb{Z}} \varphi_0(\rho - j) \in C^\infty(\mathbb{R})$$

which satisfies $\varphi_1 \ge 1$. Note that $\varphi_1(\rho - k) = \varphi_1(\rho)$ for $k \in \mathbb{Z}$, we define $\varphi(\rho) := \frac{\varphi_0(\rho)}{\varphi_1(\rho)}$, which implies

$$\sum_{j \in \mathbb{Z}} \varphi(\rho - j) = 1.$$

Let $\psi(r) := \varphi(\frac{\log r}{\log 2}) \in C_c^\infty((0,\infty))$, then

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j} r) = \sum_{j \in \mathbb{Z}} \varphi\left(\frac{\log r}{\log 2} - j\right) = 1.$$

Define $\psi_0(r) = 1 - \sum_{j=0}^{\infty} \psi(2^{-j} r)$, one can easily check that $\psi_0(r) = \begin{cases} 1, & r < \frac{1}{2} \\ 0, & r > 1 \end{cases}$, which implies our desired formula. ∎

Definition 1.35 (Fourier multiplier)**.** *Suppose $a \in L^\infty(\mathbb{R}^n)$, $u \in \mathscr{S}$, then we define*

$$a(D)u := \mathcal{F}^{-1}\left(a(\xi)\widehat{u}(\xi)\right),$$

where $D = \frac{1}{i}\partial_x$. Furthermore, suppose $\phi \in \mathscr{S}(\mathbb{R}^n)$, we can define the multiplier of tempered distributions:

$$\phi(D)u := \mathcal{F}^{-1}\left(\phi(\xi)\widehat{u}(\xi)\right) \in \mathscr{S}', \quad u \in \mathscr{S}'.$$

Now, with a slight abuse of notation, we denote

$$\psi_0(\xi) = \psi_0(|\xi|), \psi(\xi) = \psi(|\xi|),$$

which implies

$$u = \psi_0(D)u + \sum_{j=1}^{\infty} \psi(2^{-j} D)u, \quad u \in \mathscr{S}', \tag{1.1}$$

which is called the **Littlewood-Paley decomposition**.

Rather than write 2^{-j} all the time, we will write $h = 2^{-j}$ for h being a small number, representing low frequencies here.

Lemma 1.36. *Suppose $\chi \in C_c^\infty(\mathbb{R}^n)$, then for $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$\|\chi(hD)u\|_\infty \le C h^{-\frac{n}{p}} \|u\|_p, \quad \|\chi(hD)u\|_p \le C\|u\|_p,$$

where $1 \le p \le \infty$ and the constant $C = C(n,p,\chi)$ is independent of h.

Proof. We compute

$$\chi(hD)u(x) = \mathcal{F}^{-1}\left(\chi(h\xi)\widehat{u}(\xi)\right)$$
$$=\frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\chi(h\xi)u(y)\,dy\,d\xi = \frac{1}{(2\pi h)^n}\int \widehat{\chi}(-\frac{x-y}{h})u(y)\,dy.$$

The first inequality: This implies

$$|\chi(hD)u(x)| \leq \frac{C}{h^n}\|\widehat{\chi}(\frac{\cdot}{h})\|_q\|u\|_p \leq \frac{C}{h^n}h^{\frac{n}{q}}\|\widehat{\chi}\|_q\|u\|_p = Ch^{-\frac{n}{p}}\|u\|_p.$$

In fact, this inequality holds in general for $u \in L^p$. Note that for $u \in \mathscr{S}'$, $\chi(h\xi)\widehat{u}(\xi)$ is a compactly supported distribution. Since $\mathcal{F} : \mathcal{E}' \to C^\infty$, (in fact, the Fourier transform of compactly supported distribution is smooth and analytic,) we know that $\chi(hD)u$ is a well-defined smooth function, not merely a distribution. By density arguments, take $u_n \in \mathscr{S}$, such that $u_n \overset{L^p}{\to} u$, then $u_n \overset{\mathscr{S}'}{\to} u$, and furthermore, $\chi(hD)u_n \overset{\mathscr{S}'}{\to} \chi(hD)u$. And since $\chi(hD)u_n$ is a Cauchy sequence in L^∞, it converges to a unique L^∞ function $v \in L^\infty$, and then $\chi(hD)u_n \overset{\mathscr{S}'}{\to} v$. Hence, $v = \chi(hD)u$ a.e. as functions.

The second inequality: By Young's inequality,

$$\|\chi(hD)u(x)\|_p \leq \frac{1}{(2\pi h)^n}\|\widehat{\chi}(\frac{\cdot}{h})\|_1\|u\|_p \leq \frac{1}{(2\pi)^n}\|\widehat{\chi}\|_1\|u\|_p.$$

□

Theorem 1.37. *Let $u \in L^p$ where $1 \leq p \leq \infty$. Then*

$$u \in C^{0,\gamma}(\mathbb{R}^n) \Leftrightarrow \forall \chi \in C_c^\infty(\mathbb{R}^n \setminus \{0\}), \|\chi(hD)u\|_\infty \leq Ch^\gamma$$

where C depends on χ and $\gamma \in (0,1)$.

Proof. **Implication ⇒:** We compute

$$\chi(hD)u(x) = \frac{1}{(2\pi h)^n}\int \widehat{\chi}(-\frac{x-y}{h})u(y)\,dy = \frac{1}{(2\pi)^n}\int \widehat{\chi}(y)u(x+hy)\,dy$$
$$= \frac{1}{(2\pi)^n}\int \widehat{\chi}(y)(u(x+hy)-u(x))\,dy,$$

where we use the assumption $\chi(0) = 0$. Then

$$|\chi(hD)u(x)| \leq C\|u\|_{C^{0,\gamma}}\int |\widehat{\chi}(y)||hy|^\gamma\,dy \leq Ch^\gamma.$$

Implication ⇐: Let

$$\Lambda_\gamma(u) := \sup_{0<h<1} h^{-\gamma}\left(\|\psi(hD)u\|_\infty + \max_{1\leq k\leq n}\|\psi_k(hD)u\|_\infty\right),$$

where $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ as in the Littlewood-Paley decomposition (1.1). And we set $\psi_k(\xi) := \xi_k\psi(\xi)$. Since $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\|u\|_p + \Lambda_\gamma(u) \text{ is finite.}$$

Then it suffices to prove

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\left(\|u\|_p + \Lambda_\gamma(u)\right). \tag{1.2}$$

We consider the L^∞ part in $C^{0,\gamma}$ norm: From the Littlewood-Paley decomposition (1.1), we know

$$\|u\|_\infty \le \|\psi_0(D)u\|_\infty + \sum_{j=0}^{\infty} \|\psi(2^{-j}D)u\|_\infty \le \|\psi_0(D)u\|_\infty + \sum_{j=0}^{\infty} 2^{-j\gamma}\Lambda_\gamma(u) \le \|u\|_p + (1-2^{-\gamma})^{-1}\Lambda_\gamma(u),$$

where we use the definition of Λ_γ in the second inequality, and Lemma 1.36 in the last inequality.

Now we use the Littlewood-Paley decomposition (1.1) again,

$$u(x) - u(y) = \psi_0(D)u(x) - \psi_0(D)u(y) + \sum_{j=0}^{\infty} \left(\psi(2^{-j}D)u(x) - \psi(2^{-j}D)u(y)\right).$$

Let $|x-y| = r$. In order to prove (1.2), it suffices to prove the following two claims:

$$|\psi_0(D)u(x) - \psi_0(D)u(y)| \le C\|u\|_p r^\gamma, \tag{1.3}$$

$$|\psi(2^{-j}D)u(x) - \psi(2^{-j}D)u(y)| \le C\Lambda_\gamma(u) r^\gamma. \tag{1.4}$$

For (1.3), we compute

$$|\psi_0(D)u(x) - \psi_0(D)u(y)| \le \sup\left(\nabla(\psi_0(D)u)\right) r \le \frac{r}{(2\pi)^n}\int |\nabla\widehat{\psi_0}(x-y)||u(y)|\,dy \le Cr\|\nabla\widehat{\psi_0}\|_q\|u\|_p,$$

which implies (1.3) when $r \le 1$.

For (1.4), the proof is a little tricky. We will prove two estimates.

Higher frequency estimates:

$$|\psi(hD)u(x) - \psi(hD)u(y)| \le 2\|\psi(hD)u\|_\infty \le 2h^\gamma\Lambda_\gamma(u).$$

Lower frequency estimates:

$$\begin{aligned}|\psi(hD)u(x) - \psi(hD)u(y)| \le& Cr\max_k \|D_{x_k}(\psi(hD)u)\|_\infty = Crh^{-1}\max_k \|hD_{x_k}(\psi(hD)u)\|_\infty \\ =& Crh^{-1}\max_k \|\psi_k(hD)u\|_\infty \le Crh^{\gamma-1}\Lambda_\gamma(u).\end{aligned}$$

Now, the sum can be estimated as

$$\sum_{j=0}^{\infty} |\psi(2^{-j}D)u(x) - \psi(2^{-j}D)u(y)| \le C\Lambda_\gamma(u)\left(\sum_{2^j\le s} r2^{-j(\gamma-1)} + \sum_{2^j>s} 2^{-j\gamma}\right) \le \widetilde{C}\Lambda_\gamma(u)\left(rs^{1-\gamma} + s^{-\gamma}\right),$$

then choose s such that $s = \frac{1}{r}$, which leads to the desired estimate (1.4). And this completes the proof. □

Now we turn to the proof of Morrey's inequality.

Proof of Morrey's inequality. Thanks to (1.2) in the proof of the previous lemma, it suffices to prove

$$\Lambda_\gamma(u) \le C\|\nabla u\|_p, \tag{1.5}$$

where $\gamma = 1 - \frac{n}{p}$. Then by Lemma 1.36, we have

$$\|\varphi(hD)hD_{x_j}u\|_p \le Ch^{1-\frac{n}{p}}\|\nabla u\|_p$$

for any $\varphi \in C_c^\infty$. Let $\varphi_j(\xi) = \xi_j\varphi(\xi)$, then $\varphi_j(hD) = hD_{x_j}\varphi(hD)$. Hence,

$$\|\varphi_j(hD)u\|_p \le Ch^\gamma\|\nabla u\|_p, \tag{1.6}$$

where $\gamma = 1 - \frac{n}{p}$. Recall that $\Lambda_\gamma(u) = \sup_{0<h<1} h^{-\gamma}\left(\|\psi(hD)u\|_\infty + \max_k \|\psi_k(hD)u\|_\infty\right)$, it suffices to prove

$$\|\psi(hD)u\|_\infty \le Ch^\gamma \|\nabla u\|_p.$$

Now we intend to write $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ as a combination of compactly supported function of the following form

$$\psi(\xi) = \sum_j \xi_j \chi_j(\xi)$$

with $\chi_j \in C_c^\infty$. This is a precalculus problem. We choose $\chi_j(\xi) = \frac{\xi_j}{|\xi|^2}\psi(\xi)$ and this is smooth since $0 \notin \operatorname{supp}\psi$. Hence,

$$\|\psi(hD)u\|_\infty \le \sum_j \|hD_{x_j}\chi_j(hD_{x_j})u\|_\infty \le Ch^\gamma \|\nabla u\|_p,$$

where the last inequality follows from (1.6). And this completes our proof. □

Remark 1.38. Similar methods can be used to obtain regularity of solutions to PDEs: Suppose $u \in L^1(U), \Delta u = f \in C^{k,\gamma}(U), 0 < \gamma < 1$, where U is bounded. Then we have $u \in C^{k+2,\gamma}(V)$, where $V \Subset U$. See [9, Section 7.5.3]. If Δ is replaced by a differential operator P whose coefficients are $C^{k,\gamma}$, the statement also holds. This is useful to show the regularities of Variational Problems.

1.6. **Compactness theorems.** Now we turn to compactness theorems.

Definition 1.39. *Let B be a Banach space and $B' \subset B$ is another Banach space. We call $B' \subset B$ is compactly embedded (a compact inclusion) if bounded sets in B' are precompact in B and the inclusion map is continuous. That is, if $\forall \{u_n\}_{n=1}^\infty \subset B'$, $\|u_n\|_{B'} \le C$, then there exists a subsequence $n_k \to \infty$ and $u \in B$ such that*

$$\|u_{n_k} - u\|_B \overset{k\to\infty}{\to} 0;$$

and

$$\|\cdot\|_{B'} \le C\|\cdot\|_B.$$

Example 1.40. *Let $B = C([-1,1]), B' = C^1([-1,1])$ with $\|u\|_B = \sup_{|x|\le 1}|u(x)|, \|u\|_{B'} = \sup_{|x|\le 1}|u(x)| + \sup_{|x|\le 1}|u'(x)|$. If $\|u_n\|_{C^1([-1,1])} \le M$, then by the mean value theorem, we have*

$$|u_n(x)| \le M, |u_n(x) - u_n(y)| \le M|x-y|.$$

Finally, there exists $u \in C([-1,1])$, $n_k \to \infty$ such that

$$\|u_{n_k} - u\|_{C([-1,1])} \to 0$$

which follows from Arzela-Ascoli theorem.

Example 1.41 (Subexample)**.** *Let $u_n(x) = \begin{cases} |x|, & |x| > \frac{1}{n}, \\ \frac{n}{2}x^2 + \frac{1}{2n}, & |x| \le \frac{1}{n}, \end{cases}$ then one can observe that $u_n \in C^1([-1,1])$ and $\|u_n\|_{C^1([-1,1])} \le 2$. There exists $n_k = k$, $u(x) = |x| \in C^0([-1,1]) \setminus C^1([-1,1])$, $u_{n_k} \to u$ in C^0.*

Remark 1.42. Recall that if $\{u : \|u\|_B \le 1\} \subset B$ is compact for a Banach space B, then B is finite dimensional.

However, we can have a space $B' \subset B$ and $\{u \in B' : \|u\|_{B'} \le 1\} \subset B$ is compact in B even though B' is of infinite dimension. Let $B = L^q(U), 1 \le q < p^*$, $B' = W^{1,p}(U)$.

Theorem 1.43 (Rellich-Kondrachov Theorem)**.** *For $U \Subset \mathbb{R}^n, \partial U$ is C^1. Then the unit ball in $W^{1,p}(U)$ is compact in $L^q(U)$ for $1 \le q < p^*$.*

Remark 1.44. Although Gagliardo-Nirenberg-Sobolev inequality tells us $W^{1,p}(U)$ is in $L^q(U)$ when $q = p^*$, the theorem tells us in weaker space, i.e. $q < p^*$, we have stronger statement.

And note that the boundedness of U in the assumption is essential for Rellich-Kondrachov Theorem. One need to add some decay assumptions if U is unbounded.

Before we give a proof of Rellich-Kondrachov Theorem 1.43 for the special case $p = 2$, we recall some useful theorems:

Definition 1.45 (Dual space)**.** *Let B be an Banach space. Then*

$$B^* := \{u : B \to \mathbb{C} : |u(x)| \le C\|x\|_B,\ \forall x \in B\}$$

is its dual space with norm $\|u\|_{B^*} = \sup_{\|x\|_B = 1} |u(x)|$.

Theorem 1.46 (Poisson summation formula in dimension 1)**.** *Let $a \in \mathbb{R}$, $a \neq 0$, we have*

$$\sum_{k\in\mathbb{Z}} e^{ikax} = \frac{2\pi}{a} \sum_{k\in\mathbb{Z}} \delta(x - \frac{2\pi}{a}k), \tag{1.7}$$

in the sense of distributions, that is, for all $\varphi \in \mathscr{S}(\mathbb{R})$,

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi}(ka) = \frac{2\pi}{a} \sum_{k\in\mathbb{Z}} \varphi(\frac{2\pi}{a}k). \tag{1.8}$$

Proof. Note that both sides of (1.8) converge for $\varphi \in \mathscr{S}$. And by the multiply and divide trick analogous to that in Example 1.17, we know both sides of (1.7) are well-defined tempered distributions. We compute

$$(1 - e^{iax}) \sum_{k\in\mathbb{Z}} e^{ikax} = \sum_{k\in\mathbb{Z}} e^{ikax} - \sum_{k\in\mathbb{Z}} e^{i(k+1)ax} = 0$$

in the sense of distributions. Let $w(x) = \sum_{k\in\mathbb{Z}} e^{ikax}$, then this implies $-2ie^{-\frac{iax}{2}} \sin(\frac{ax}{2}) w(x) = 0$, which tells us supp$w \subset \{\frac{2\pi}{a}k\}_{k\in\mathbb{Z}}$. Moreover, since $\sin(\frac{ax}{2})$ vanishes simply at these points, we have

$$w(x) = \sum_{k\in\mathbb{Z}} c_k \delta(x - \frac{2\pi}{a}k).$$

Since $e^{ikax} = e^{ika(x+\frac{2\pi}{a})}$, we know $w(x + \frac{2\pi}{a}) = w(x)$, which implies $c_k = c(a)$ is independent of k. Now we proved that for all $\varphi \in \mathscr{S}$,

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi}(ka) = c(a) \sum_{k\in\mathbb{Z}} \varphi(\frac{2\pi}{a}k).$$

We replace $\varphi(\cdot)$ by $\varphi(\cdot + x)$, we get

$$\sum_{k\in\mathbb{Z}} e^{ikax}\widehat{\varphi}(ka) = c(a)\sum_{k\in\mathbb{Z}} \varphi(\frac{2\pi}{a}k + x).$$

Assume $\varphi \in C_c^\infty((0, \frac{2\pi}{a}))$ and then integrate both sides from 0 to $\frac{2\pi}{a}$, we have

$$\frac{2\pi}{a}\widehat{\varphi}(0) = c(a)\int_0^{\frac{2\pi}{a}} \varphi(x)\,dx = c(a)\widehat{\varphi}(0),$$

which implies $c(a) = \frac{2\pi}{a}$. □

Analogously, Poisson summation formula holds for dimension n as follows.

Theorem 1.47 (Poisson summation formula in dimension n)**.** *Let $a \in \mathbb{R}$, $a \neq 0$, we have*

$$\sum_{k\in\mathbb{Z}^n} e^{iak\cdot x} = \left(\frac{2\pi}{a}\right)^n \sum_{k\in\mathbb{Z}^n} \delta(x - \frac{2\pi}{a}k).$$

Take $a = 1$ in Poisson summation formula and pair both sides with $\varphi(\cdot + x)$ for $\varphi(\cdot) \in \mathscr{S}$, we have

Corollary 1.48.

$$\frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(k)e^{ik\cdot x} = \sum_{k\in\mathbb{Z}^n} \varphi(x - 2\pi k)$$

where $\varphi \in \mathscr{S}$.

In particular, for $\varphi \in C_c^\infty((-\pi, \pi)^n)$, we have

$$\frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(k)e^{ik\cdot x} = \varphi(x).$$

Then by density arguments, we have

Corollary 1.49 (Fourier series characterization of L^2 norm of compact support L^2 functions)**.**

$$\frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n} \widehat{u}(k)\overline{\widehat{v}(k)} = \int u(x)\overline{v(x)}\,dx$$

for $u, v \in L^2$, $\operatorname{supp} u, \operatorname{supp} v \subset (-\pi, \pi)^n$. In particular,

$$\|u\|_{L^2}^2 = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n} |\widehat{u}(k)|^2$$

for $u \in L^2$ and $\operatorname{supp} u \subset (-\pi, \pi)^n$.

Now we give a proof of Rellich-Kondrachov Theorem 1.43 for the special case $p = 2$ which is different from that in the book [2]:

Proof of Rellich-Kondrachov Theorem when $p = 2$. Let $\overline{U} \subset B(0, R)$ and we assume WLOG that $R = 1$. For any $\|v_n\|_{H^1(U)} \leq 1$, there exists $u_n \in H^1(\mathbb{R}^n)$, such that $u_n|_U = v_n$, $\|u_n\|_{H^1(\mathbb{R}^n)} \leq 1$, $\operatorname{supp} u \subset B(0, 1)$. Now we need to find a subsequence u_{n_k} that is a Cauchy sequence in L^2, then $u_{n_k} \overset{k\to\infty}{\to} u$ in L^2.

We claim that for $w \in H^1(\mathbb{R}^n)$ and $\text{supp} w \subset B(0,1)$, then

$$\frac{1}{C}\sum_{k\in\mathbb{Z}^n}\langle k\rangle^2|\widehat{w}(k)|^2 \le \|w\|_{H^1}^2 \le C\sum_{k\in\mathbb{Z}^n}\langle k\rangle^2|\widehat{w}(k)|^2$$

Thanks to $w \in L^2$ and $\partial^\alpha w \in L^2$, we can apply Corollary 1.49 and get $\|w\|_{L^2}^2 = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n}|\widehat{w}(k)|^2$, $\|\partial^\alpha w\|_{L^2}^2 = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n}|k^\alpha\widehat{w}(k)|^2$ for $|\alpha|=1$. In fact,

$$\|w\|_{H^1}^2 = \int(1+|\xi|^2)|\widehat{w}|^2\,dx = \int|w(x)|^2+|\nabla w(x)|^2\,dx = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n}\langle k\rangle^2|\widehat{w}(k)|^2, \tag{1.9}$$

for all $w\in H^1, \text{supp} w \subset (-\pi,\pi)^n$. Hence, for u_n, we have $\|u_n\|_{H^1(\mathbb{R}^n)}^2 = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n}\langle k\rangle^2|\widehat{u_n}(k)|^2 \le 1$.

We define $\Pi_p : L^2([-\pi,\pi]^n) \to \mathbb{C}^{N_p}$ (N_p is finite), with $\Pi_p w = (\{\widehat{w}(l)\})_{|l|\le p}$ and

$$\Pi_p w(x) := \sum_{|l|\le p}\widehat{w}(l)e^{il\cdot x}.$$

Here we claim our key estimate

$$\|(I-\Pi_p)w\|_{L^2}^2 \le \langle p\rangle^{-2}\|w\|_{H^1}^2,$$

which follows from

$$\sum_{|l|>p}|\widehat{w}(l)|^2 = \sum_{|l|>p}\langle l\rangle^{-2}\langle l\rangle^2|\widehat{w}(l)|^2 \le \langle p\rangle^{-2}\|w\|_{H^1}^2.$$

Now we want to find $\{n_k\}$ such that $\|u_{n_k}-u_{n_l}\|_{L^2}\to 0$ as $k,l\to\infty$.

Step 1: For all p, $\|\Pi_p u_n\|_{\mathbb{C}^{N_p}} \le \|u_n\|_{L^2} \le \|u_n\|_{H^1} \le 1$. Since bounded closed sets in $\mathbb{C}^{N_p}$ are compact, we can choose $\{n_k^{p+1}\}_{\in\mathbb{N}} \subset \{n_k^p\}_{k\in\mathbb{N}}$ successively such that $\Pi_p u_{n_k^p}$ converges in $\mathbb{C}^{N_p}$ for every p. And

$$\limsup_{k,l\to\infty}\|u_{n_k^p}-u_{n_l^p}\|_{L^2} \le 2\langle p\rangle^{-2}$$

since $\|u_{n_k^p}-u_{n_l^p}\|_{L^2} \le \|\Pi_p u_{n_k^p}-\Pi_p u_{n_l^p}\|_{L^2} + \|(I-\Pi_p)(u_{n_k^p}-u_{n_l^p})\|_{L^2} \le \|\Pi_p u_{n_k^p}-\Pi_p u_{n_l^p}\|_{L^2} + 2\langle p\rangle^{-2}$.

Step 2: Choose $n_k = n_k^k$, then for $k<l$, $n_l \in \{n_m^k\}_{m\in\mathbb{N}}$, which implies

$$\limsup_{k,l\to\infty}\|u_{n_k}-u_{n_l}\| = 0.$$

Hence, $\{u_{n_k}\}_{k\in\mathbb{Z}}$ is Cauchy in L^2. □

Remark 1.50. We prove this in a very hands-on way without using Acsoli-Arzela theorem so that we can see the mechanism here: Compactness means that you can reduce it to finite dimensions modulo something small. And then you can use that something small to make the tail go to zero.

1.7. **Final comments on Sobolev spaces.**

Theorem 1.51 (Poincare's inequality Version 1)**.** *Assume $U \Subset \mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1\le p<n, 1\le q\le p^*$, then*

$$\|u\|_{L^q(U)} \le C\|\nabla u\|_{L^p(U)}.$$

Proof. By the Gagliardo-Nirenberg-Sobolev inequality,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for $u \in C_c^\infty(U)$. Then

$$\|u\|_{L^q(U)} \le \|u\|_{L^{p^*}(U)} \le C\|\nabla u\|_{L^p(U)}.$$

Since $W^{1,p}(U) = \overline{C_c^\infty(U)}^{W^{1,p}}$, the desired result follows. ∎

Theorem 1.52 (Poincare's inequality Version 2)**.** *Assume $U \Subset \mathbb{R}^n$. For any $1 \le p < \infty$ and suppose $u \in W_0^{1,p}(U)$, then we have*

$$\|u\|_{L^p(U)} \le C\|\nabla u\|_{L^p(U)}.$$

Proof. **Case $p < n$:** This is immediate from the first version of Poincare's inequality.

Case $n \le p < \infty$: Let $q = n - \varepsilon$. Note that $q^* = \frac{n(n-\varepsilon)}{\varepsilon}$, we chooose $\varepsilon \ll 1$ such that $q^* \ge p$. We apply the first version of Poincare's inequality to get

$$\|u\|_{L^p(U)} \le C\|\nabla u\|_{L^q(U)} \le C\|\nabla u\|_{L^p(U)}.$$

∎

Remark 1.53. Your enemy is the constant function, which has zero gradient. However, the zero trace condition eliminates this possibility.

Theorem 1.54 (Riesz representation theorem for Hilbert space)**.** *Let H be a Hilbert space. If $\Phi : H \to \mathbb{C}$ such that $|\Phi(u)| \le C\|u\|$, then there exists $v \in H$ such that*

$$\Phi(u) = \langle u, v\rangle_H.$$

Theorem 1.55. *$H^{-s}(\mathbb{R}^n)$ is the dual space to $H^s(\mathbb{R}^n)$ in the following sense: if $v \in H^{-s}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$, then $\langle u, v\rangle_{L^2} := \int u\overline{v}$ is well-defined if $u, v \in \mathscr{S}(\mathbb{R}^n)$. And $\forall \Phi : H^s(\mathbb{R}^n) \to \mathbb{C}$, $|\Phi(u)| \le C\|u\|_{H^s}$, there exists $v \in H^{-s}$ such that $\Phi(u) = \langle u, v\rangle_{L^2}$. (Note that $\langle \cdot, \cdot\rangle_{L^2}$ here can also be viewed as the distributional pairing.)*

Proof. We first assume $u, v \in \mathscr{S}$. Then

$$\int u\overline{v} = (2\pi)^{-n}\int \widehat{u}(\xi)\overline{\widehat{v}(\xi)} = (2\pi)^{-n}\int \langle\xi\rangle^s\widehat{u}(\xi)\langle\xi\rangle^{-s}\overline{\widehat{v}(\xi)},$$

which implies

$$|\langle u, v\rangle_{L^2}| \le (2\pi)^{-n}\|u\|_{H^s}\|v\|_{H^{-s}}.$$

By the density of $\mathscr{S}$ in H^s and H^{-s}, $\int u\overline{v}$ can be defined by approximation.

Now suppose Φ as above. Theorem 1.54(Riesz representation theorem) tells us there exists $w \in H^s$ such that

$$\Phi(u) = \langle u, w\rangle_{H^s} = (2\pi)^{-n}\int \langle\xi\rangle^{2s}\widehat{u}(\xi)\overline{\widehat{w}(\xi)}.$$

We define v as

$$\widehat{v}(\xi) := \langle\xi\rangle^{2s}\widehat{w}(\xi).$$

Obviously, $v \in H^{-s}$ and $\Phi(u) = \langle u, v\rangle_{L^2}$. ∎

When it comes to bounded domains, duality is trickier. Recall that $H_0^1(U) = \{u \in H^1(U) : Tu = 0\} = \overline{C_c^\infty(U)}^{H^1}$, we define

Definition 1.56. *Let $U \Subset \mathbb{R}^n$, then*

$$H^{-1}(U) = \{u \in \mathcal{D}'(U) : \forall \varphi \in C_c^\infty(U), |u(\varphi)| \le C\|\varphi\|_{H^1}\}.$$

with the norm $\|u\|_{H^{-1}(U)} = \sup\{|u(\varphi)| : \varphi \in H_0^1(U), \|\varphi\|_{H_0^1} \le 1\}$.

Equivalently,

$$H^{-1}(U) = \{u \in \mathcal{D}'(U) : \forall \varphi \in C_c^\infty(U) : |u(\varphi)| \le C\|\nabla\varphi\|_{L^2}\}$$

by Poincare's inequality for $H_0^1(U)$.

Example 1.57. *For the distributional derivative $\partial_{x_j} : L^2 \subset \mathcal{D}'(U) \to \mathcal{D}'(U)$, we have the property that $\partial_{x_j} : L^2 \to H^{-1}$. Since,*

$$|\partial_{x_j} u(\varphi)| = |u(\partial_{x_j}\varphi)| \le \|u\|_{L^2}\|\partial_{x_j}\varphi\|_{L^2}, \forall \varphi \in C_c^\infty(U),$$

we know $\partial_{x_j} u \in H^{-1}$ by definition.

Theorem 1.58. *More generally, $u \in H^{-1}(U)$ if and only if*

$$\exists u_0, u_1, \cdots, u_n \in L^2(U), \text{ such that } u = u_0 + \sum_{j=1}^n \partial_{x_j} u_j.$$

Example 1.59. *In 1 dimensional case, we have*

$$H_0^1((0,\pi)) = \{u = \sum_{n=1}^\infty a_n \sin nx : \sum_n |a_n|^2 n^2 < \infty\}.$$

Note that $H^1 \subset C^0$ by Morrey's inequality when $n = 1$. One can consider the odd extension of $u \in H_0^1((0,\pi))$, then from (1.9), we know it can be represented by sine series. By duality, we have the same type of characterization

$$H^{-1}((0,\pi)) = \{v = \sum_{n=1}^\infty b_n \sin nx (\text{with convergence only in } \mathcal{D}') : \sum_n |b_n|^2 n^{-2} < \infty\},$$

where the formal series $\sum b_n \sin nx$ only converge when it pairs with distributions.

In higher dimension, the analogous result

$$H_0^1((0,\pi)^n) = \{u = \sum_{l \in \mathbb{N}_*^n} a_l \sin l_1 x_1 \cdots \sin l_k x_k : \sum_{l \in \mathbb{N}_*^n} |l|^2 |a_l|^2 < \infty\}$$

also holds.

Remark 1.60. We have already used same trick when we dealt with heat equation with zero boundary condition:

$$\begin{cases} (\partial_t - \partial_x^2)u = 0, \\ u(0,x) = f(x), \\ u(t,0) = u(t,\pi) = 0, t > 0. \end{cases}$$

Let $f(x) = \sum_{n=1}^{\infty} a_n \sin nx \in L^2$. (This can be done since C_c^∞ is dense in L^2, hence we can form Fourier series with sine terms only.) Then $u(t,x) = \sum_{n=1}^{\infty} e^{-tn^2} a_n \sin nx \in C_0^\infty((0,\pi)), t > 0$.

2. Calculus of Variations

Example 2.1. *One would like to find a function $y = f(x)$ such that*

$$f(a) = c, f(b) = d, \tag{2.1}$$

and the graph of f has shortest length. The length of the graph is

$$L(f) = \int_a^b (1 + f'(x)^2)^{\frac{1}{2}} \, dx.$$

We want to minimize it over all functions f satisfying (2.1)*. If f is a minimizer, then for $\forall \varphi \in C_c^\infty((a,b))$, $t \mapsto L(f + t\varphi)$ has to have a minimizer at $t = 0$. We compute*

$$\frac{d}{dt} L(f + t\varphi) = \int_a^b \frac{\varphi'(x)(f'(x) + t\varphi'(x))}{(1 + (f'(x) + t\varphi'(x))^2)^{\frac{1}{2}}} \, dx,$$

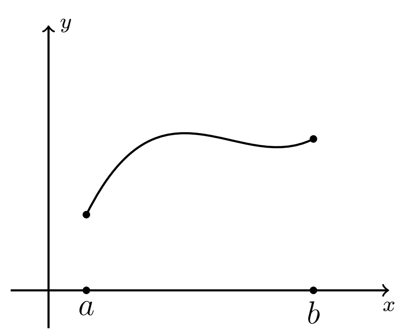
when f is a "nice" function, then we have

$$0 = \frac{d}{dt}\Big|_{t=0} L(f + t\varphi) = \int_a^b \varphi'(x) \frac{f'(x)}{(1 + f'(x)^2)^{\frac{1}{2}}} \, dx, = -\int_a^b \varphi(x) \left(\frac{f'(x)}{(1 + f'(x)^2)^{\frac{1}{2}}} \right)' dx,$$

for all $\varphi \in C_c^\infty((a,b))$. Hence,

$$\frac{d}{dx} \left(\frac{f'(x)}{(1 + f'(x)^2)^{\frac{1}{2}}} \right) = 0,$$

which is equivalent to $\frac{f'(x)}{(1+f'(x)^2)^{\frac{1}{2}}} = C$, and finally, $f'(x) = \widetilde{C}$. Thus, $f(x) = \alpha x + \beta$.



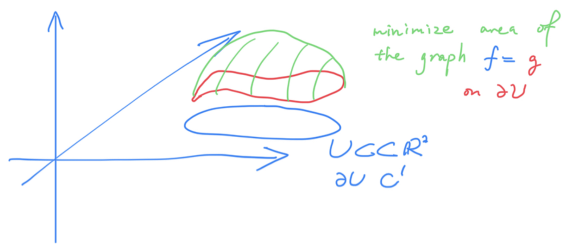
Example 2.2. *The area can be expressed as*

$$A(f) = \iint_U (1 + |\nabla f(x)|^2)^{\frac{1}{2}} \, dx,$$

over all f satisfies $f = g$ on ∂U. If f is a minimizer, then for $\forall \varphi \in C_c^\infty(U)$, $t \mapsto A(f + t\varphi)$ has to have a minimizer at $t = 0$. We calculate again and derive that f satisfies

$$div\left(\frac{\nabla f}{(1+|\nabla f|^2)^{\frac{1}{2}}}\right) = 0,$$

which is indeed the minimal surface equation.



Remark 2.3. Though we will not solve this, we demonstrate the idea we will use later. Try $f \in H^1(U)$, $Tf = g \in L^2(\partial U)$, $A(f) \geq 0$. Let $m := \inf\{A(f) : f \in H^1(U), f|_{\partial U} = g\}$, which implies $\exists f_j \in H^1(U), f_j|_{\partial U} = g$ such that $A(f_j) \to m$. Could we find $f_{j_k} \to f$, which is the minimizer? The answer is no for this example. Intuitively, we can add tentacles on the surface while the change of area is very small. And we will make stronger assumptions on our functionals which will guarantee that we can find such sequences. But those assumptions will not be satisfied for the minimal surface equation.

Example 2.4. *Given any two circles, what is the minimal surface whose boundary is exactly these two circles? (The surface need not to be connected.)*

When the two circles are far apart, then the minimal surface will be two flat disks. However, if they are close to each other, the minimal surface will be the catenoid.

2.1. **General Setup.** Assume $U \Subset \mathbb{R}^n$. Let $L : \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$ be a C^∞ function, and we write it as $L(p, z, x)$. For the gradient, we write as $D_pL = (\partial_{p_1}L, \cdots, \partial_{p_n}L)$. Let $I[w] = \int_U L(Dw(x), w(x), x)\,dx$ and we want to mimimize this among functions such that $w|_{\partial U} = g$ where g is a prescribed function.

Now suppose w is a minimizer, then for all $\varphi \in C_c^\infty(U)$, we have

$$0 = \int_U \frac{d}{dt}\left(L(Dw + tD\varphi, w + t\varphi, x)\right)\,dx\Big|_{t=0} = \int_U \left(D\varphi \cdot D_pL(Dw, w, x) + \varphi D_zL(Dw, w, x)\right)\,dx$$
$$= \int_U \left(-\sum_{j=1}^n \left(L_{p_j}(Dw, w, x)\right)_{x_j} + D_zL(Dw, w, x)\right)\varphi(x)\,dx,$$

which implies

$$-\sum_{j=1}^n \left(L_{p_j}(Dw, w, x)\right)_{x_j} + D_zL(Dw, w, x) = 0. \tag{2.2}$$

And this is called the Euler-Lagrange equation.

Example 2.5. *Let* $L(p,z,x) = \frac{1}{2}|p|^2 - f(x)z$*, then* $I[w] = \int_U \left(\frac{1}{2}|\nabla w|^2 - f(x)w(x)\right)\,dx$*. And the corresponding Euler-Lagrange equation is*

$$-\sum (w_{x_j})_{x_j} - f(x) = 0,$$

which is the same as

$$-\Delta w = f,\ \ w|_{\partial U} = g.$$

Hence, the Poisson equation with Dirichlet boundary condition can be solved if we find a minimizer for the functional $I[w]$*.*

There are two natural generalizations. The first direction is that we can make it nonlinear by letting $L(p,z,x) = \frac{1}{2}|p|^2 + F(z)$ *and* $f(t) = F'(t)$*. Then the Euler-Lagrange equation is*

$$-\Delta w = f(w).$$

The other direction is to have non-constant coefficients. One can consider $L(p,z,x) = \frac{1}{2}\sum_{i,j} a_{ij}p_ip_j + \sum_j b_j(x)p_j(x) - f(x)z$*, where* $a_{ij} = a_{ji}$*. Then the Euler-Lagrange equation is*

$$-\sum_{i,j=1}^{n} \partial_{x_j}\left(a_{ij}(x)\partial_{x_i}w(x)\right) = f(x).$$

This is nice when $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$*,* $\forall \xi \in \mathbb{R}^n$ *and* $\forall x \in \overline{U}$*.*

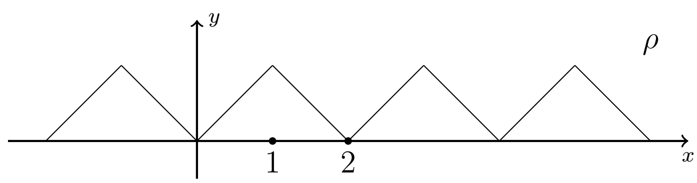
2.2. **Second derivative test.** Let $i(t) = I[w + t\varphi]$ with $\varphi \in C_c^\infty$. If we have a local minimizer w, then $i'(0) = 0, i''(0) \geq 0$. Since

$$i''(0) = \frac{d}{dt}\Big|_{t=0}\int_U \left(\sum_j \varphi_{x_j}\partial_{p_j}L(Dw + tD\varphi, w + t\varphi, x) + \varphi\partial_z L(Dw + tD\varphi, w + t\varphi, x)\right)dx$$

$$= \int_U \left(\sum_{i,j}\varphi_{x_i}\varphi_{x_j}\partial^2_{p_ip_j}L(Dw,w,x) + 2\sum_j \varphi\varphi_{x_j}\partial_z\partial_{p_j}L(Dw,w,x) + \varphi^2\partial_z^2 L(Dw,w,x)\right)dx \tag{2.3}$$

is nonnegative. In fact, (2.3) makes sense for compactly supported φ that is merely lipschitz continuous such that supp$\varphi \subset U$. One can see by taking a convolution with a standard mollifier $\eta \in C_c^\infty$. To be specific, by the results in [2, Appendix C.5], set $\varphi^\varepsilon = \eta_\varepsilon * \varphi$, then $\nabla\varphi^\varepsilon \to \nabla\varphi$ and $|\nabla\varphi^\varepsilon|$ is uniformly bounded for all ε, which implies that (2.3) holds for φ by replace φ by φ^ε in (2.3) and then letting $\varepsilon \to 0$.

Take $\varphi(x) = \varepsilon\rho(\frac{x\cdot\xi}{\varepsilon})\zeta(x)$, where $\rho(t) = \begin{cases} t, & t \in [0,1] \\ 2-t, & t \in [1,2] \end{cases}$ and $\rho(t+2) = \rho(t)$.



Using this φ, we have

$$\varphi_{x_j}(x) = \varepsilon\rho(\frac{x\cdot\xi}{\varepsilon})\zeta_{x_j}(x) + \xi_j\rho'(\frac{x\cdot\xi}{\varepsilon})\zeta(x) = \xi_j\rho'(\frac{x\cdot\xi}{\varepsilon})\zeta(x) + O(\varepsilon).$$

Then $i''(0) \geq 0$ implies

$$0 \leq \int_U \left(\sum_{i,j} \xi_i \xi_j \partial^2_{p_i p_j} L \right) |\rho'|^2 \zeta^2 \, dx + O(\varepsilon).$$

Since $|\rho'(t)| = 1$ almost everywhere, by letting $\varepsilon \to 0$, we have

$$\int_U \left(\sum_{i,j} \xi_i \xi_j \partial^2_{p_i p_j} L \right) \zeta^2 \, dx \geq 0$$

for all $\zeta \in C_c^\infty(U)$. Moreover, for any $\xi \in \mathbb{R}^n$,

$$\sum_{i,j} \xi_i \xi_j \partial^2_{p_i, p_j} L(Dw(x), w(x), x) \geq 0.$$

Hence, it is useful to assume convexity:

$$p \mapsto L(p, z, x) \text{ is convex,}$$

and when L is smooth, convexity is equivalent to the positive definiteness of second derivatives

$$\sum_{i,j=1}^n \xi_i \xi_j \partial^2_{p_i p_j} L(p, z, x) \geq 0, \ \forall \xi \in \mathbb{R}^n, (p, z, x) \in \mathbb{R}^n \times \mathbb{R} \times \overline{U}.$$

Note that another useful equivalent characterization of convexity is

$$L(q) \geq L(p) + (q - p) \cdot D_p L(p).$$

Definition 2.6 (Strict convexity)**.** *We say $p \mapsto L(p, z, x)$ is strictly convex if*

$$\sum_{i,j=1}^n \xi_i \xi_j \partial^2_{p_i p_j} L(p, z, x) \geq c|\xi|^2, \ \forall \xi \in \mathbb{R}^n, (p, z, x) \in \mathbb{R}^n \times \mathbb{R} \times \overline{U}, c > 0.$$

Example 2.7. *Suppose $L = \frac{1}{2} \sum a_{ij} p_i p_j$, where $a_{ij} = a_{ji}$. Then strict convexity means that*

$$\sum_{ij} a_{ij}(x) \xi_i \xi_j \geq c|\xi|^2.$$

Moreover, the corresponding Euler-Lagrange equation is

$$-\sum_{i,j} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} w \right) = 0.$$

Then the strict convexity implies the differential operator in the Euler-Lagrange equation

$$Pw = -\sum_{i,j} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} w \right)$$

is elliptic.

Example 2.8. *For the minimal surface equation,* $L = (1+|p|^2)^{\frac{1}{2}}$*, then* $L_{p_i} = \frac{p_i}{(1+|p|^2)^{\frac{1}{2}}}$ *and* $L_{p_ip_j} = \frac{\delta_{ij}}{(1+|p|^2)^{\frac{1}{2}}} - \frac{p_ip_j}{(1+|p|^2)^{\frac{3}{2}}}$*. Hence,*

$$\sum_{ij} \xi_i\xi_j L_{p_ip_j} = \frac{1}{(1+|p|^2)^{\frac{3}{2}}}\left(|\xi|^2(1+|p|^2) - \sum_{ij} p_ip_j\xi_i\xi_j\right) = \frac{|\xi|^2(1+|p|^2) - \langle \xi, p\rangle^2}{(1+|p|^2)^{\frac{3}{2}}},$$

which goes to 0 *as* $|p| \to \infty$*. Thus, it is not strict convex.*

2.3. **Existence of minimizers.** The first condition is a lower bound of L, which is called coercivity.

Definition 2.9 (Coercivity)**.** *We say* L *satisfies the coercivity condition if*

$$\exists \alpha > 0, \beta \geq 0, s.t.\ L(p,z,x) \geq \alpha|p|^q - \beta,\ \forall z \in \mathbb{R}, x \in \overline{U},$$

where $1 < q < \infty$*.*

This gives us the following bound

$$I[w] \geq \alpha\|Dw\|_q^q - \beta m(U).$$

So we can always assume $\beta = 0$ by replacing L by $L+\beta$. Let $\mathcal{A} = \{w \in W^{1,q}(U) : u|_{\partial U} = g\}$ and we will minimize $I[w]$ over $\mathcal{A}$. Note that the boundedness of $I[w]$ will imply the boundedness of the gradient of w. And this will imply compactness in L^p for $p < q^*$, which implies some kind of convergence.

The second condition is trickier.

Definition 2.10 (Lower semicontinuity in the weak sense)**.** *We say* $I[\cdot]$ *satisifes weakly lower semicontinuity in* $W^{1,q}(U)$ *if for any* $u_k \rightharpoonup u$ *in* $W^{1,q}$*, that is,* $u_k \rightharpoonup u$ *in* L^q*,* $Du_k \rightharpoonup Du$ *in* L^q*. we have* $I[u] \leq \liminf I[u_k]$*.*

Before we establish the existence theorem (Theorem 2.21), one can recall some basic facts.

Theorem 2.11 (Characterization of weak convergence in $W^{1,q}(U)$)**.** *The following two statements are equivalent:*
(1) $u_k \rightharpoonup u$ *in* $W^{1,q}(U)$*;*
(2) $u_k \rightharpoonup u$ *in* L^q*,* $Du_k \rightharpoonup Du$ *in* L^q*.*

Proof. **(1)⇒(2):** For all $v \in L^{q'}(U) = (L^p(U))^*$, we know from the Holder's inequality that $g \mapsto \int gv$, $g \mapsto \int (Dg)v$ are bounded linear functional on $W^{1,q}(U)$, respectively. Hence, $\int u_kv \to \int uv$, $\int (Du_k)v \to \int (Du)v$, which implies the weak convergence of u_k, Du_k in L^q.

(2)⇒(1): Now we view $W^{1,q}(U)$ as a subspace of $L^p(U;\mathbb{R}^{n+1})$ by the isometry map $v \mapsto (v, Dv)$. Then for any $\phi \in (W^{1,q}(U))^*$, we can extend it as a bounded linear functional on $L^p(U;\mathbb{R}^{n+1})$ by Hahn-Banach theorem, also denoted by ϕ, that is, $\phi = (\phi_j)_{j=0}^n \in (L^p(U;\mathbb{R}^{n+1}))^* = L^q(U,\mathbb{R}^{n+1})$. Then from our assumption, we have $\int \phi_0 u_k \to \int \phi_0 u$ and $\int \phi_j D_{x_j}u_k \to \int \phi_j D_{x_j}u$, $j = 1, \cdots, n$. And this implies

$$\phi(u_k) = (\int \phi_0 u_k, \int \phi_1 D_{x_1}u, \cdots, \int \phi_n D_{x_n}u) \to \phi(u),$$

which completes the proof. ∎

Theorem 2.12 (Banach-Alaoglu theorem)**.** *The unit ball $\{u \in B^* : \|u\|_{B^*} \le 1\}$ is weak* compact. Equivalently, suppose $\|u_n\|_{B^*} \le M$, then there exists n_k, $u \in B^*$, such that*

$$u_{n_k}(x) \overset{*}{\rightharpoonup} u(x)$$

for all $x \in B$, i.e. u_{n_k} is of weak convergence.*

Corollary 2.13. *If B is reflexive, that is, $(B^*)^* = B$, then $\{x : \|x\|_B \le 1\}$ is weak compact.*

Example 2.14. *We can set $B = L^q(U), 1 < q < \infty$ in our corollary.*

Notation 2.15. *Let $\{x_j\} \subset B$, then $x_j \rightharpoonup x \in B$ if and only if $u(x_j) \to u(x)$ for all $u \in B^*$.*

Theorem 2.16. *Suppose B is reflexive, $x_j \rightharpoonup x$, then $\|x\|_B \le \liminf \|x_j\|_B$.*

Proof. This follows from

$$|x(u)| = \lim |x_j(u)| = \liminf |x_j(u)| \le \liminf \|x_j\|_B \|u\|_{B^*}$$

and $\|x\|_B = \sup_{\|u\|_{B^*}=1} |x(u)|$. □

Theorem 2.17. *Suppose B is reflexive, $x_j \rightharpoonup x$, then there exists C such that $\sup_j \|x_j\|_B \le C$.*

Proof. Since $\forall u \in B^*$, $|x_j(u)| \le C(u)$. By Banach-Steinhaus theorem, $\sup_j \|x_j\|_B \le C$. □

Then we have a corollary:

Theorem 2.18. *Suppose $w_j \rightharpoonup w$ in L^q, then $\sup_j \|w_j\|_{L^q} \le C$. Moreover, $\|w\|_{L^q} \le \liminf_{j\to\infty} \|w_j\|_{L^q}$.*

Theorem 2.19 (Special case of Mazur's theorem)**.** *Suppose $V \subset B$ is a closed subspace, then V is weakly closed.*

Proof. Suppose $x_j \in V$, $x_j \rightharpoonup x \in B$, that is, for all $u \in B^*$, $u(x_j) \to u(x)$. So if $x \notin V$, we want to construct $u \in B^*$, such that $u(x_j) = 0$ and $u(x) = 1$, which leads to a contradiction.

Let $\widetilde{V} = V + \mathbb{C}x$ with $\widetilde{\varphi} : \widetilde{V} \to \mathbb{C}$ defined by $\widetilde{\varphi}(y + \alpha x) = \alpha$ for $y \in V, \alpha \in \mathbb{C}$. Now we prove that there exists some constant $C > 0$, for all $y \in V, \alpha \in \mathbb{C}$, $|\widetilde{\varphi}(y + \alpha x)| \le C\|y + \alpha x\|_B$ by contradiction. Suppose not, then for all n, there exists y_n, α_n, $|\alpha_n| = |\widetilde{\varphi}(y_n + \alpha_n x)| > n\|y_n + \alpha_n x\|_B$, which is equivalent to $\frac{1}{n} > \|\frac{y_n}{\alpha_n} + x\|_B$. This means that $V \ni -\frac{y_n}{\alpha_n} \to x \in V$, which is impossible since V is closed. So we get a contradiction. In other words, we can extend $\widetilde{\varphi}$ by Hahn-Banach theorem to $u \in B^*$ such that $u|_{\widetilde{V}} = \widetilde{\varphi}$. □

Theorem 2.20 (Convexity implies weakly lower semicontinuity)**.** *Suppose $L \ge -C$ and $p \mapsto L(p, z, x)$ is convex for all $(z, x) \in \mathbb{R} \times \overline{U}$. Then for any $1 < q < \infty$, $w \mapsto I[w]$ is weakly lower semicontinuous in $W^{1,q}(U)$.*

This theorem makes it easier to verify weak lower semicontinuity. Before we give the proof of this theorem, we can use this to establish the existence theorem for the minimizer.

Theorem 2.21 (Existence of minimizers)**.** *Suppose $p \mapsto L(p,z,x)$ is convex for all $(z,x) \in \mathbb{R} \times \overline{U}$ and L satisfies the coercivity condition. Suppose that $\mathcal{A} = \{w \in W^{1,q}(U) : w|_{\partial U} = g\} \neq \varnothing$ with $g \in L^q(\partial U)$, then there exists $u \in \mathcal{A}$ such that $I[u] = \min_{w\in\mathcal{A}} I[w]$.*

Proof. We can assume without loss of generality that $\beta = 0$. Put $m = \inf_{w\in\mathcal{A}} I[w] \neq \infty$. Choose a sequence $u_k \in \mathcal{A}$ such that $I[u_k] \to m$. Then $I[u_k] \geq \alpha |Du_k|^q$, which implies that $\|Du_k\|_{L^q} \leq C$.

Fix $w \in \mathcal{A}$, then $u_k - w \in W_0^{1,q}(U)$, which allows us to use the Poincare inequality (Theorem 1.52) to get

$$\|u_k\|_{L^q} \leq \|u_k - w\|_{L^q} + \|w\|_{L^q} \leq \|Du_k - Dw\|_{L^q} + \|w\|_{L^q} \leq C + \|w\|_{W^{1,q}}.$$

Hence, $\|u_k\|_{W^{1,q}}$ is uniformly bounded. Apply Banach-Alaoglu theorem to $\{u_k\} \subset L^q$ and $\{Du_k\} \subset L^q$ respectively, we extract a subsequence, also denoted by u_k, such that $u_k \rightharpoonup u$ in $W^{1,q}$. This means that $u_k - w \rightharpoonup u - w$ in $W^{1,q}$. Since $u_k - w \in W_0^{1,q}(U)$, a closed subspace of $W^{1,q}(U)$, by the special case of Mazur's theorem (Theorem 2.19), we know $u - w \in W_0^{1,q}(U)$, which implies $u \in \mathcal{A}$.

Now from the convexity and Theorem 2.20, I is weakly lower semicontinuous. Hence, $I[u] \leq \liminf I[u_k] = m$. Thus $u \in \mathcal{A}$ is a minimizer. □

Now it's time to give a proof of Theorem 2.20. The proof is a little involved.

Proof of Theorem 2.20. Let $u_j \rightharpoonup u$ in L^q, $Du_j \rightharpoonup Du$ in L^q and $l = \liminf_k I[u_k]$. In the following proof, we will taking subsequence many many times without changing notation.

By taking a subsequence, $l = \lim_k I[u_k]$. Since weak convergence implies $\|u_k\|_{L^q} \leq C, \|Du_k\|_{L^q} \leq C$, and using Theorem 1.43, we know $W^{1,q}(U)$ is compact in $L^q(U)$, we know $u_k \to u$ in L^q by taking a subsequence. Hence, $u_k \to u$ almost everywhere by taking a subsequence by applying Riesz-fischer theorem.

And now we can apply Egorov theorem: For any $\varepsilon > 0$, there exists E_ε such that $m(U \setminus E_\varepsilon) < \varepsilon$ and u_k converges uniformly to u on E_ε provided that $m(U) < \infty$. Let $F_\varepsilon = \{x \in U : |u(x)| + |Du(x)| \leq \frac{1}{\varepsilon}\}$. Then $m(U \setminus F_\varepsilon) \to 0$ as $\varepsilon \to 0$ since $u, Du \in L^q(U)$ implies $\{|u(x)| = \infty \text{ or } |Du(x)| = \infty\}$ is of measure zero. Let $G_\varepsilon = E_\varepsilon \cap F_\varepsilon$, then $m(U \setminus G_\varepsilon) \to 0$. Without loss of generality, assuming $L \geq 0$ by adding C to L. Now

$$I[u_k] = \int_U L(Du_k, u_k, x) \geq \int_{G\varepsilon} L(Du_k, u_k, x) \geq \int_{G\varepsilon} L(Du, u_k, x) + D_pL(Du, u_k, x)(Du_k - Du),$$

where the last inequality follows from convexity.

Since Du is bounded on G_ε and $u_k \to u$ uniformly on G_ε, we know $L(Du, u_k, x) \to L(Du, u, x)$ uniformly on G_ε. Then one integrate it over a set of finite measure and get the convergence

$$\lim_k \int_{G_\varepsilon} L(Du, u_k, x) \to \int_{G_\varepsilon} L(Du, u, x).$$

For the second term, we write

$$\begin{aligned} D_pL(Du, u_k, x)(Du_k - Du) = (D_pL(Du, u_k, x) - D_pL(Du, u, x))\,(Du_k - Du) \\ + D_pL(Du, u, x)(Du_k - Du). \end{aligned}$$

Since $Du_k - Du$ are bounded in L^q, $D_pL(Du, u_k, x) - D_pL(Du, u, x) \to 0$ uniformly on G_ε,

$$\begin{aligned}
&\left|\int_{G_\varepsilon} (D_pL(Du, u_k, x) - D_pL(Du, u, x))\,(Du_k - Du)\right| \\
\leq &\sup_{G_\varepsilon} |D_pL(Du, u_k, x) - D_pL(Du, u, x)|\,\|Du_k - Du\|_{L^1(G_\varepsilon)} \\
\leq &\sup_{G_\varepsilon} |D_pL(Du, u_k, x) - D_pL(Du, u, x)|\,\|Du_k - Du\|_{L^q(G_\varepsilon)} \\
\leq &C\sup_{G_\varepsilon} |D_pL(Du, u_k, x) - D_pL(Du, u, x)| \to 0.
\end{aligned}$$

Moreover, $D_pL(Du, u, x)$ is bounded on G_ε and $Du_k \rightharpoonup Du$ in L^q. Since $L^\infty(G_\varepsilon) \subset L^{q'}(G_\varepsilon)$,

$$\int_{G_\varepsilon} D_pL(Du, u, x)(Du_k - Du) \to 0.$$

Hence,

$$l = \lim I[u_k] \geq \int_{G_\varepsilon} L(Du, u, x), \quad \forall \varepsilon > 0.$$

Since $L \geq 0$, we have

$$l \geq \lim_{\varepsilon} \int_{G_\varepsilon} L(Du, u, x) = \int_U L(Du, u, x) = I[u]$$

which follows from the monotone convergence theorem. □

Example 2.22. *Let $L(p, z, x) = \sum_{i,j=1}^n a_{ij}(x)p_ip_j$, where $a_{ij}(x) = a_{ji}(x)$ and the quadratic form L is strictly convex, i.e. $a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$, $\forall \xi \in \mathbb{R}^n, x \in \overline{U}$.*

In this case, we have the coercivity condition for $q = 2$. From the last problem from the first problem set, we know

$$v \mapsto v|_{\partial_{\mathbb{R}^n_+}}$$

is a surjective bounded map from $H^1(\mathbb{R}^n) \to H^{\frac{1}{2}}(\mathbb{R}^{n-1})$. Then $\mathcal{A} = \{u \in H^1(U) | u|_{\partial U} = g\} \neq \varnothing$ for any $g \in H^{\frac{1}{2}}(U)$ Though the formulation is somewhat different. but if you straighten out boundary, it's all local. Since U is compact, you get the result.

2.4. **Uniqueness of minimizers.**

Theorem 2.23. *Suppose $L = L(p, x)$ independent of z and there exists $\theta > 0$ such that for all $\xi \in \mathbb{R}^n, p \in \mathbb{R}^n, x \in \overline{U}$,*

$$\sum_{i,j} L_{p_ip_j}(p, x)\xi_i\xi_j \geq \theta|\xi|^2,$$

which means L is uniformly strictly convex. Then any minimizer of $I[u]$ among $\mathcal{A} = \{w \in W^{1,q}(U) : w|_{\partial U} = g\} \neq \varnothing$ is unique.

Proof. Let $u, \widetilde{u}$ be minimizers of $I[\cdot]$, $v = \frac{u+\widetilde{u}}{2}$. Then by Taylor's formula, uniformly strictly convexity implies

$$L(p, x) \geq L(q, x) + D_pL(q, x) \cdot (p - q) + \frac{\theta}{2}|p - q|^2.$$

We apply this formula with $p = Du, q = D\widetilde{u}$, then

$$m = I[u] \geq I[v] + \int \left(D_pL\left(\frac{Du + D\widetilde{u}}{2}, x\right) \cdot \frac{Du - D\widetilde{u}}{2} + \frac{\theta}{8}(Du - D\widetilde{u})^2 \right) dx.$$

For $I[\widetilde{u}]$, we also have

$$m = I[\widetilde{u}] \geq I[v] + \int \left(D_pL\left(\frac{Du + D\widetilde{u}}{2}, x\right) \cdot \frac{D\widetilde{u} - Du}{2} + \frac{\theta}{8}(Du - D\widetilde{u})^2 \right) dx.$$

Then we add them together and get

$$2m \geq 2I[v] + \frac{\theta}{4}\int (Du - D\widetilde{u})^2 \, dx \geq 2m + \frac{\theta}{4}\int (Du - D\widetilde{u})^2 \, dx,$$

which implies $\int |Du - D\widetilde{u}|^2 \, dx$. Hence, $Du = D\widetilde{u}$ almost everywhere. But $u|_{\partial U} = \widetilde{u}|_{\partial U}$, we have $u = \widetilde{u}$ almost everywhere. $\square$

2.5. **Weak solution of an elliptic operator in divergence form.**

Example 2.24. *Let $L(p, x) = \sum_{i,j} a_{ij}(x)p_ip_j \geq \theta|p|^2$, $a_{ij}(x) = a_{ji}(x)$, $\theta > 0$. Then by computing the derivative of $I[u+t\varphi]$ at $t = 0$, we know that the corresponding Euler-Lagrange equation*

$$\sum_{i,j=1^n} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) = 0, u|_{\partial U} = g \in H^{\frac{1}{2}}(U)$$

is solved weakly (in the sense of distributions) and u is unique in $H^1(U)$.

Now we consider

$$\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) = f, \; u|_{\partial U} = g \in H^{\frac{1}{2}}(U), \; f \in H^{-1}(U), \tag{2.4}$$

where $\sum_{i,j} a_{ij}(x)p_ip_j \geq \theta|p|^2$, $a_{ij}(x) = a_{ji}(x)$. Let $I[w] = \int_U \sum_{i,j} a_{ij}(x)\partial_{x_i}w\partial_{x_j}w - f(x)u \, dx$. Though $L(p, z, x) = \sum_{i,j} a_{ij}(x)p_ip_j - f(x)z$ does not satisfy the coercivity condition, we do not need $L \geq \alpha|p|^2 - \beta$ to be satisfied for all p, z since z and p are related each other in the actual integral. We only need $I[w] \geq \int_U |Dw|^2 - \beta$.

Now we assume $f \in L^2(U)$ at first.

Lemma 2.25 (Peter-Paul inequality)**.** *For all $\varepsilon > 0$,*

$$2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2.$$

It means that "you pay Peter to rob Paul a lot".

Fix $w_0 \in H^1(U)$ with $w_0|_{\partial U} = g$. Using the Peter-Paul inequality, we have

$$\begin{aligned} I[w] &\geq \int_U \left(\theta|Dw|^2 - |f||w|\right) dx \geq \int_U \left(\theta|Dw|^2 - \frac{\varepsilon}{2}|w|^2 - \frac{1}{2\varepsilon}|f|^2\right) dx \\ &\geq \int_U \left(\theta|Dw|^2 - \frac{\varepsilon}{2}\left(|w - w_0|^2 + |w_0|^2\right) - \frac{1}{2\varepsilon}|f|^2\right) dx \qquad (2.5) \\ &\geq \int_U \left(\theta - \frac{C\varepsilon}{2}\right)|Dw|^2 \, dx - C(\varepsilon, w_0, f), \end{aligned}$$

where $w - w_0 \in H_0^1$. Choose $\varepsilon < \frac{2\theta}{C}$ such that $\alpha = \theta - \frac{C\varepsilon}{2} > 0$, we have $I[w] \ge \alpha \int_U |Dw|^2 - \beta$. Moreover, thanks to the convexity of L in p, we can apply Theorem 2.21 to know there is a solution u to this problem when $f \in L^2$.

Now we need to remedy $f \in L^2$. We consider this problem

$$\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) = f \in H^{-1}(U),\ u|_{\partial U} = 0. \tag{2.6}$$

Since the solution $u \in H_0^1(U)$,

$$|\int fu| \le \|f\|_{H^{-1}}\|u\|_{H_0^1} \le \frac{1}{\varepsilon}\|f\|_{H^{-1}} + \varepsilon\|u\|_{H_0^1(U)} \le \frac{1}{\varepsilon}\|f\|_{H^{-1}} + \varepsilon\|Du\|_{L^2(U)},$$

where $\int fu$ is meant as distributional pairing since $H^{-1}(U)$ is the dual of $H_0^1(U)$. Then we can prove an analogous estimate like (2.5), which allows us to use the same argument conclude that there exists a solution to (2.6) (in the weak sense) with the desired regularities. And since we have already solved when the right hand side is zero in Example 2.24, we finally solved (2.4) (in the weak sense) by adding the two solutions together.

The discussion above implies the following theorem.

Theorem 2.26. *There exists a unique solution $u \in H^1(U)$ (in the weak sense) of the elliptic equation*

$$\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) = f,\ u|_{\partial U} = g \in H^{\frac{1}{2}}(U),\ f \in H^{-1}(U),$$

with $\sum_{i,j} a_{ij}(x)p_ip_j \ge \theta|p|^2$, $a_{ij}(x) = a_{ji}(x)$.

Proof. The existence follows from the discussion above. Now we prove the uniqueness. Suppose not, then $v = u - \widetilde{u} \in H_0^1(U)$ solves

$$\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}v\right) = 0$$

weakly. By the definition of weak solution, we multiply this by v and get

$$0 = \sum_{i,j=1}^{n} \int_U a_{ij}(x)\partial_{x_j}v\partial_{x_i}v.$$

However, by the ellipticity of (a_{ij}),

$$\sum_{i,j=1}^{n} \int_U a_{ij}(x)\partial_{x_j}v\partial_{x_i}v \ge \theta \int_U |Dv|^2,$$

which implies $Dv = 0$ almost everywhere in U. Hence, $v = 0$. ∎

2.6. **Weak solutions of Euler-Lagrange equation.** Assume

$$\begin{aligned} |L(p,z,x)| &\le C(|p|^q + |z|^q + 1), \\ |D_pL(p,z,x)| &\le C(|Du|^{q-1} + |u|^{q-1} + 1), \\ |D_zL(p,z,x)| &\le C(|Du|^{q-1} + |u|^{q-1} + 1). \end{aligned} \tag{2.7}$$

for some constant C and all $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in \overline{U}$. Then

$$\begin{aligned}|\partial_{p_j} L(Du,u,x)| \leq& C(|Du|^{q-1}+|u|^{q-1}+1) \in L^{q'}(U),\\ |\partial_{z} L(Du,u,x)| \leq& C(|Du|^{q-1}+|u|^{q-1}+1) \in L^{q'}(U),\end{aligned}$$

provided $u \in W^{1,q}(U)$.

Definition 2.27. *Suppose the assumption (2.7) holds and $u \in \mathcal{A} = \{w \in W^{1,q}(U) : w|_{\partial U} = g\}$. We then say that the Euler-Lagrange equation holds weakly if for all $u \in W_0^{1,q}(U)$,*

$$\int_U \left(\sum_{j=1}^n \partial_{p_j} L(Du,u,x) v_{x_j} + \partial_z L(Du,u,x) v\right) = 0. \tag{2.8}$$

Theorem 2.28. *Suppose $u \in \mathcal{A}$ is a minimizer for L satisfying (2.7). Then u is a weak solution to the Euler-Lagrange equation, that is, (2.8) holds for all $v \in W_0^{1,q}(U)$.*

Proof. Let $i(t) = I[u+tv]$, $v \in W_0^{1,q}(U)$. Set

$$L^t(x) := \frac{L(Du+tDv, u+tv, x) - L(Du,u,x)}{t},$$

then $L^t(x) \to \sum_j \partial_{p_j} L(Du,u,x) v_{x_j} + \partial_z L(Du,u,x)v$ almost everywhere in x. Since

$$\frac{1}{t}\left(i(t) - i(0)\right) = \int_U L^t(x)\, dx,$$

we want to bound L^t by an L^1 function so that we can get the conclusion from the Dominated Convergence Theorem. From the fundamental theorem of calculus, we know

$$\begin{aligned}&|L(Du+tDv, u+tv, x) - L(Du,u,x)|\\ =&\int_0^t \sum_j \left|\partial_{p_j} L(Du+sDv, u+sv, x)\right| \left|v_{x_j}\right| + \left|\partial_z L(Du+sDv,u+sv,x)\right| |v|\, ds\\ \leq&\int_0^t \left(|Du+sDv|^{q-1} + |u+sv|^{q-1} + 1\right)\left(|Dv| + |v|\right) ds\\ \leq& Ct\left(|Du|^{q-1} + |Dv|^{q-1} + |u|^{q-1} + |v|^{q-1}\right)\left(|Dv|+|v|\right)\\ \leq& Ct\left(|Du|^{q-1}(|Dv|+|v|) + |u|^{q-1}(|Dv|+|v|) + |Dv|^q + |v|^q + 1\right)\\ \leq& Ct\left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1\right),\end{aligned}$$

where we use Young's inequality $ab \leq \frac{a^q}{q} + \frac{b^{q'}}{q'}$ in the last step. Since $|Du|^q + |u|^q + |Dv|^q + |v|^q + 1 \in L^1(U)$ is independent of t, we know $\lim_{t\to 0} \frac{i(t)-i(0)}{t} = \int_U \lim_{t \to 0} L^t(x)\, dx$. Hence,

$$0 = i'(0) = \int_U \left(\sum_j \partial_{p_j} L(Du,u,x) v_{x_j} + \partial_z L(Du,u,x) v\right).$$

□

Now we focus on under what circumstances the converse is true.

Theorem 2.29. *Suppose $u \in \mathcal{A}$ is a weak solution to the Euler-Lagrange equation. If $(p, z) \mapsto L(p, z, x)$ is convex for all $x \in \overline{U}$, then u is a minimizer of $I[\cdot]$.*

Proof. Covexity implies

$$L(p, z, x) + D_p L(p, z, x) \cdot (q - p) + D_z L(p, z, x) \cdot (w - z) \leq L(q, w, x).$$

For all $v \in \mathcal{A}$, we integrate this with $p = Du(x), q = Dv(x), z = u(x), w = v(x)$, then

$$I[u] + \int_U D_p L(Du, u, x) \cdot (Dv - Du) + D_z L(Du, u, x) \cdot (v - u)\, dx \leq I[v].$$

But $v - u \in W_0^{1,q}(U)$ and u satisfies the equation weakly, which tells us $I[u] \leq I[v]$ from the inequality above. $\square$

2.7. **Regularities of weak solutions.** Can weak solution be upgraded to strong solutions? We consider the simplest case

$$L(q, z, x) = \frac{1}{2}|p|^2 - zf(x)$$

with $U \Subset \mathbb{R}^n, \partial U$ is C^∞. Suppose $f \in C^\infty(\overline{U}), g \in C^\infty(\partial U)$. which is much better than the assumption in Theorem 2.26, then we have $u \in C^\infty(\overline{U})$.

We will show $u \in H^2$ in the general case and $C^\infty(\overline{U})$ (full regularity) only in the linear case. It is extremely difficult to prove the full regularity in the nonlinear case so that we will not cover that here.

In this part, we make some stronger assumptions that

$$L(p, z, x) = L(p) - zf(x), \;\; f \in L^2(U) \tag{2.9}$$

with $|L(p)| \leq C(|p|^2 + 1)$, $|D_p L(p)| \leq C(|p| + 1)$, $|D_p^2 L(p)| \leq C$. Here $q = 2$ when compared to the assumptions in the previous part. Finally, we assume uniformly convexity: $\sum_{i,j} L_{p_i p_j} \xi_i \xi_j \geq \theta |\xi|^2$ for all $p \in \mathbb{R}^n, \xi \in \mathbb{R}^n$.

Theorem 2.30. *Let $u \in H_0^1(U)$ satisfies the Euler-Lagrange equation corresponding to (2.9) weakly. Then there exists some constant C such that*

$$\|u\|_{H^1} \leq C\|f\|_{L^2}.$$

Proof. We use the definition of weak solution with $v = u \in H_0^1(U)$. Namely,

$$\int_U \sum_j L_{p_j}(Du) u_{x_j} = \int_U f(x) u(x)\, dx.$$

The uniformly convexity implies $(DL(p) - DL(0)) \cdot p \geq \theta |p|^2$. We use this with $p = Du$, then

$$\int_U f(x) u(x)\, dx = \int_U \left(D_p L(Du) \cdot Du - D_p L(0) \cdot Du\right) dx \geq \theta \int_U |Du|^2\, dx,$$

where $\int_U D_p L(0) \cdot Du\, dx = 0$ by the divergence theorem. Moreover, by Cauchy inequality, Peter-Paul inequality, Poincare inequality, we have

$$\left| \int_U f(x) u(x)\, dx \right| \leq \frac{C\varepsilon}{2} \|Du\|_{L^2} + \frac{1}{2\varepsilon} \|f\|_{L^2}.$$

Hence, if we take $\varepsilon \ll 1$ such that $\frac{C\varepsilon}{2} < \theta$ in the equality above, we get

$$\|u\|_{H^1} \le C'\|Du\|_{L^2} \le \widetilde{C}\|f\|_{L^2}.$$

□

Now we consider the interior regularity with $U \Subset \mathbb{R}^n$.

Theorem 2.31 (Interior regularity)**.** *Suppose $u \in H_0^1(U)$ satifies the Euler-Lagrange equation corresponding to* (2.9)

$$-\sum_{j=1}^n \left(L_{p_j}(Du)\right)_{x_j} = f$$

weakly with $f \in L^2$. Then $u \in H^2_{loc}(U)$, that is, for all $K \Subset U$, $u|_K \in H^2(K)$.

Proof. Take open sets $V \Subset W \Subset U$. Choose a function $\zeta \in C_c^\infty(W)$ such that $\zeta \equiv 1$ on V. We define the difference quotients as

$$D_k^h u(x) = \frac{u(x+he_k)-u(x)}{h}.$$

If h is small enough ($h \ll d(W,U^c)$), then it is well-defined on U. Note that we have the identity

$$\int_{\mathbb{R}^n} wD_k^{-h}v = -\int_{\mathbb{R}^n} vD_k^h w \tag{2.10}$$

by a change of variable. Since $-(\zeta^2 D_k^h u) \in H_0^1(W) \subset H_0^1(U)$, by the definition of weak solution, we have

$$-\int_U fD_k^{-h}\left(\zeta^2 D_k^h u\right)\, dx = -\int_U \sum_{j=1}^n L_{p_j}(Du)\ \left(D_k^{-h}\left(\zeta^2 D_k^h u\right)\right)_{x_j}\, dx$$
$$= -\int_U \sum_{j=1}^n L_{p_j}(Du)\ D_k^{-h}\left(\zeta^2 D_k^h u\right)_{x_j}\, dx = \int_U \sum_{j=1}^n D_k^h\left(L_{p_j}(Du)\right)\ \left(\zeta^2 D_k^h u\right)_{x_j}\, dx,$$

where we use the fact $D_k^h\left((\cdot)_{x_j}\right) = \left(D_k^h(\cdot)\right)_{x_j}$ in the third equality and use the identity (2.10) in the last equality with $v := \left(\zeta^2 D_k^h u\right)_{x_j}$, $w := L_{p_j}(Du)$. Note that for $v : \mathbb{R}^n \to \mathbb{R}^n$, we have

$$D_k^h\left[F(v(\cdot))\right] = \frac{1}{h}\left(F(v(x+he_k)) - F(v(x))\right) = \frac{1}{h}\int_0^1 D_s F\left(sv(x+he_k)+(1-s)v(x)\right)\, ds$$
$$= \frac{1}{h}\int_0^1 \sum_{i=1}^n F_{p_i}\left(sv(x+he_k)+(1-s)v(x)\right)\cdot\left(v_i(x+he_k)-v_i(x)\right).$$

Hence, use this formula with $v = Du$, and combine the two formulas above, we get

$$\begin{aligned}-\int_U fD_k^{-h}\left(\zeta^2 D_k^h u\right)\, dx &= \frac{1}{h}\int_U \sum_{i,j=1}^n a_{ij}^h(x)\left(u_{x_i}(x+he_k)-u_{x_i}(x)\right)\left(\zeta^2 D_k^h u\right)_{x_j}\, dx \\ &= \int_U \sum_{i,j=1}^n a_{ij}^h(x) D_k^h u_{x_j}(x)\left(\zeta^2 D_k^h u\right)_{x_j}\, dx = I_1 + I_2,\end{aligned} \tag{2.11}$$

where $a_{ij}^h(x) := \int_0^1 L_{p_ip_j}(sDu(x+he_k)+(1-s)Du(x))\,ds$, and

$$I_1 := \int_U \sum_{i,j=1}^n a_{ij}^h(x) D_k^h u_{x_j}(x) D_k^h u_{x_i} \zeta^2\,dx, \quad I_2 := \int_U \sum_{i,j=1}^n 2a_{ij}^h(x) D_k^h u_{x_j}(x)\zeta\zeta_{x_i} D_k^h u.$$

By the uniform convexity, we know

$$I_1 \geq \theta \int_U \zeta^2 |D_k^h Du(x)|^2\,dx,$$

and

$$|I_2| \leq C\int_W \zeta |D_k^h Du||D_k^h u|\,dx \leq \varepsilon \int_W \zeta^2 |D_k^h Du(x)|^2\,dx + \frac{C}{\varepsilon}\int_W |D_k^h u|^2\,dx.$$

Furthermore, thanks to Lemma 2.33, the left hand side of (2.11) can be bounded as follows:

$$\begin{aligned}
&\left| -\int_U f D_k^{-h}\left(\zeta^2 D_k^h u\right)\,dx \right| \leq \int_U |f||D_k^{-h}(\zeta^2 D_k^h u)| \\
\leq& \varepsilon \int_U |D_k^{-h}(\zeta^2 D_k^h u)|^2\,dx + \frac{C}{\varepsilon}\int_U |f|^2\,dx \leq \varepsilon \int_U |D(\zeta^2 D_k^h u)|^2\,dx + \frac{C}{\varepsilon}\int_U |f|^2\,dx \\
\leq& \varepsilon \int_W \zeta^2 |D_k^h Du|^2\,dx + \varepsilon \int_W |D_k^h u|^2\,dx + \frac{C}{\varepsilon}\int_U |f|^2\,dx \\
\leq& \varepsilon \int_U \zeta^2 |D_k^h Du|^2\,dx + \frac{C}{\varepsilon}\left(\int_U |Du|^2 + |f|^2\,dx\right).
\end{aligned}$$

Hence, choose $\varepsilon \ll 1$, then we have

$$\varepsilon \int_U \zeta^2 |D_k^h Du|^2\,dx + \frac{C}{\varepsilon}\left(\int_U |Du|^2 + |f|^2\,dx\right) \geq (\theta-\varepsilon)\int_U \zeta^2 |D_k^h Du(x)|^2\,dx - \frac{C}{\varepsilon}\int_U |D_k^h u|^2\,dx,$$

which implies

$$\int_V |D_k^h Du(x)|^2\,dx \leq \int_U \zeta^2 |D_k^h Du(x)|^2\,dx \leq C' \int_U |Du|^2 + |f|^2\,dx.$$

Finally, we get $D_k^h Du$ is bounded in $L^2(V)$ for all h, so by Banach-Alaoglu theorem, we know $\left\{D_k^h Du\right\}_{h>0}$ is weakly compact. Hence, by passsing to a subsequence, we know $D_k^{h_j} Du \rightharpoonup v \in L^2(V)$. It follows that

$$\int v\varphi\,dx \leftarrow \int D_k^{h_j} Du\varphi\,dx = \int Du D_k^{-h_j}\varphi \to -\int Du\varphi_{x_k}\,dx,$$

for all $\varphi \in C_c^\infty(V)$, which implies that $v \in L^2(V)$ is the weak derivative of Du. Thus, $u \in H^2_{loc}(U)$. □

Remark 2.32. The main idea in the proof is that you can estimate derivatives with cutoff functions if you only need local results. This requires carefully choosing the width in the difference quotients to stay away from the boundary.

Here we turn to the lemma we used to establish the estimate for (2.11) in the proof above.

Lemma 2.33 (L^p estimate for difference quotients)**.** *Suppose* $1 \le p < \infty$ *and* $u \in W^{1,p}(U)$*. Then for each* $V \Subset U$*,*

$$\|D^h u\|_{L^p(V)} \le C\|Du\|_{L^p(U)}$$

for some constant C *and all* $h \le \frac{1}{2}d(V, \partial U)$*.*

Proof. We first assume $u \in C^\infty(U) \cap W^{1,p}(U)$. Since

$$D_k^h u(x) = \frac{u(x+he_k) - u(x)}{h} = \frac{1}{h}\int_0^1 \frac{d}{dt}u(x+the_k)\,dt = \int_0^1 u_{x_k}(x+the_k)\,dt,$$

we have

$$\int_V |D_k^h u(x)|^p\,dx \lesssim \int_V \int_0^1 |u_{x_k}(x+the_k)|^p\,dt\,dx \lesssim \int_0^1 \int_V |u_{x_k}(x+the_k)|^p\,dx\,dt \lesssim \int_U |Du|^p\,dx.$$

The result for $u \in W^{1,p}(U)$ follows from approximation. □

Theorem 2.34 (Boundary regularity)**.** *Suppose* $u \in H_0^1(U)$ *satifies the Euler-Lagrange equation corresponding to* (2.9)

$$-\sum_{j=1}^n \left(L_{p_j}(Du)\right)_{x_j} = f$$

weakly with $f \in L^2$ *and* C^2 *boudnary* ∂U*. Then* $u \in H^2(U)$*.*

We only sketch the proof here. The proof consists of several steps:

(1) We consider the special case that U is a half ball, $U = B(0,1) \cap \mathbb{R}^n_+$. Then when you consider $k \le n$, $D_k^h u$ is still in H_0^1. Hence, we can perform the same type of proof for the derivatives $\partial_{x_k} Du$ with $k \le n$.

(2) For $u_{x_n x_n}$, we use the original equation:

$$\sum_j (L_{p_j}(Du))_{x_j} = L_{p_n p_n}(Du)u_{x_n x_n} + \text{ terms involving } \partial_{x_k} Du$$

where $L_{p_n p_n}(Du) > 0$ since $(L_{p_i p_j})$ is positive definiteness. Consequently, we have bound for $u_{x_n x_n}$ by $\partial_{x_k} Du$.

(3) For U is a general region, we straighten out the boundary by a map.(Need careful check.)

For higher regularity, we assume $f \equiv 0$. Take $w \in C^\infty(U)$ and put $v = -w_{x_k}$. We test against this and get

$$\sum_i \int L_{p_i}(Du)\partial_{x_k}(w_{x_i}) = 0.$$

Since $u \in H^2_{loc}$, we can do the integration by parts

$$\int \sum_{i,j} L_{p_i p_j}(Du) u_{x_j x_k} w_{x_i}\,dx.$$

Let $\widetilde{u} = u_{x_k} \in H^1$, then it satisfies

$$\sum_{i,j} \partial_{x_j}\left(a_{ij}(x)\partial_{x_i}\widetilde{u}\right) = 0$$

weakly, where $a_{ij}(x) = L_{p_i p_j}(Du) \in L^\infty$. Then $\widetilde{u} \in C^{0,\gamma}_{loc}(U)$ by a theorem from De-Giorgi Nash Moser. Then this implies $a_{ij} \in C^{0,\gamma}$, which proves $u \in C^{2,\gamma}$ by Schauder estimates.

3. Microlocal Analysis

In this section, our main reference is [4]. And we use X to denote open sets in $\mathbb{R}^n$.

We want to generalize expression like

$$\delta_0(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}} e^{ix\cdot\xi}\, d\xi,$$

which makes no sense, but it is an oscillating integral, that is, the integrand oscillates rapidly. The integral means that

$$\delta_0(\psi) = \psi(0) = \frac{1}{(2\pi)^n} \int \left(\int e^{ix\cdot\xi} \psi(x)\, dx \right) d\xi, \psi \in C_c^\infty$$

It makes sense when we test it against a function ψ. We first integrate in x and then integrate in ξ, then the integral makes sense. We will generalize this to more general integrals.

Generalization:

(1) Phase function: $x \cdot \xi, x \in \mathbb{R}^n, \xi \in \mathbb{R}^n \rightsquigarrow \varphi(x, \theta), x \in \mathbb{R}^n, \theta \in \mathbb{R}^N$, where $\varphi(x, \lambda\theta) = \lambda\varphi(x, \theta)$, $\lambda > 0$.
(2) Amplitude: $\frac{1}{(2\pi)^n} \rightsquigarrow a(x, \theta)$.
(3) Integral:

$$I(a, \varphi) = \int_{\mathbb{R}^N} a(x, \theta) e^{i\varphi(x,\theta)}\, d\theta.$$

We will discuss what condition is needed such that $I(a, \varphi) \in \mathcal{D}'(X)$.

Example 3.1 (Motivation)**.** *Let $X = \mathbb{R}^n$, $N = n$. Take $P(\xi)$ is a homogeneous polynomial of degree m such that $P(\xi) \neq 0$ when $\xi \neq 0$. Choose $\chi \in C_c^\infty$ with $\chi \equiv 1$ near 0. Let*

$$E(x) = \frac{1}{(2\pi)^n} \chi(x) \int_{\mathbb{R}^n} \frac{1 - \chi(\xi)}{P(\xi)} e^{ix\cdot\xi}\, d\xi,$$

which is a form of expression for us at this moment. Note that $\frac{1-\chi(\xi)}{P(\xi)}$ is homogeneous of degree m away from a neighborhood of 0. So if $m > n$, then the integral converges. Denote $D_x = \frac{1}{i}\partial_x$, we have

$$P(D)E(x) = \frac{1}{(2\pi)^n} \left(\chi(x) \int_{\mathbb{R}^n} \frac{1 - \chi(\xi)}{P(\xi)} P(\xi) e^{ix\cdot\xi}\, d\xi + [P(D), \chi] \int_{\mathbb{R}^n} \frac{1 - \chi(\xi)}{P(\xi)} e^{ix\cdot\xi}\, d\xi \right)$$
$$= \frac{1}{(2\pi)^n} \chi(x) \int e^{ix\cdot\xi}\, d\xi + \frac{1}{(2\pi)^n} \chi(x) \int (-\chi(\xi)) e^{ix\cdot\xi}\, d\xi + \sum_{|\alpha|>1,\beta} C_{\alpha,\beta} \partial^\alpha \chi \int \frac{(1 - \chi(\xi))}{P(\xi)} \xi^\beta e^{ix\cdot\xi}\, d\xi.$$

Note that the first term is δ_0, which we discussed before, the second term is well-defined since the integrand is compactly supported. For the third term, notice that $\partial^\alpha \chi$ is compactly supported away from 0, then we introduce

$$\frac{1}{|x|^2} \langle x, D_\xi \rangle e^{ix\cdot\xi} = e^{ix\cdot\xi}, \text{ (well-defined away from } 0\text{)}$$

to integrate by parts(if it can be justified) here many many times, then we get rapid decay with respect to ξ in the integrand of the third term. Recall that the Fourier transform of something with rapid decay is a smooth function. Hence, we get

$$P(D)E(x) = \delta_0(x) + K(x), \quad K \in C_c^\infty(\mathbb{R}^n).$$

This means that we obtained an almost fundamental solution for $P(D)$, that is, we get a distribution E, and $P(D)E$ is a delta function plus something compactly supported and smooth. Later, we will call this a parametrix for $P(D)$. The error is extremely nice, compactly supported and smooth.

It will help us to solve (approximately)

$$P(D)u = f \in \mathcal{E}'(\mathbb{R}^n).$$

Then

$$W = E * f = \int E(x-y)f(y)\,dy,$$

which is a formal way of writing, but for $f \in \mathcal{E}'$, we can make sense of it. Finally,

$$P(D)w = P(D)(E * f) = (P(D)E) * f = f + K * f,$$

where $K * f \in C^\infty(\mathbb{R}^n)$, which means that we find a solution modulo something smooth.

In this example, the amplitude $a(x,\xi) = \frac{1-\chi(\xi)}{P(\xi)}$, where $P(\lambda\xi) = \lambda^m P(\xi), \lambda > 0$ and $P(\xi) \neq 0$ if $\xi \neq 0$. Then $P(\xi) \geq c|\xi|^m$ for some $c > 0$ since P is bounded from below on $\{|\xi| = 1\}$. And by induction, we have $|\partial^\beta P(\xi)| \leq c|\xi|^{m-|\beta|}$. Thus,

$$\left|\partial_\xi^\alpha \left(\frac{1-\chi(\xi)}{P(\xi)}\right)\right| \leq C_\alpha \langle\xi\rangle^{-m-|\alpha|}.$$

This motivates the upcoming general requirements for amplitudes.

Example 3.2 (Commutator)**.** *Let $P(D) = |D|^2 = -\Delta$, where $D = \frac{1}{i}\partial$. Then*

$$[P(D),\chi]u = \chi(\Delta u) - \Delta(\chi u) = \chi(\Delta u) - \chi(\Delta u) - 2\nabla\chi \cdot \nabla u - \Delta\chi u.$$

Hence,

$$[P(D),\chi] = -2\nabla\chi \cdot -\Delta\chi.$$

Now we turn to the general theory and we focus on amplitudes first.

3.1. **Amplitudes.** Let $X \subset \mathbb{R}^n$ be an open set.

Definition 3.3 (Symbols)**.** *We say $S^m_{\rho,\delta}$ is the space of symbols of order m and of type (ρ,δ), which is defined as*

$$S^m_{\rho,\delta}(X \times \mathbb{R}^N) := \{a \in C^\infty(X \times \mathbb{R}^N)$$
$$s.t. \forall K \Subset X, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^N, \exists C = C(K,\alpha,\beta), |\partial_x^\alpha \partial_\theta^\beta a| \leq C\langle\theta\rangle^{m-\rho|\beta|+\delta|\alpha|}\}.$$

Remark 3.4. We are only interested in $0 \leq \rho \leq 1, 0 \leq \delta \leq 1$. Suppose $\rho > 1$, in that case, we claim that a has to decay rapidly in ξ, that is, $|\partial_\xi^\alpha a| \leq C_{N,\alpha}\langle\xi\rangle^{-N}$. We argue by dimensional analysis to think intuitively. Set ξ has dimension *inch*, a also has dimension *inch*^{k}, then $\partial_\xi^\alpha a$ has dimension *inch*$^{k-|\alpha|}$. However, $|\partial_\xi^\alpha a| \leq C\langle\xi\rangle^{m-\rho|\alpha|}$ which has dimension *inch*$^{m-\rho|\alpha|}$ near ∞. (We can forget about the 1 inside $\langle\xi\rangle$ since we are focusing at infinity.) Then we have

$k-|\alpha| = m-\rho|\alpha|$, that implies $k = m-(\rho-1)|\alpha|$, the units at infinity, can be any negative number, which means that it decays very very fast.

To make this rigorous, one can apply $|\theta|\partial_{|\theta|} = \sum_j \theta_j \partial_{\theta_j}$ as polar coordinates and integrate in $|\theta|$ to obtain a decay in the initial m by $(\rho-1)$. So one can obtain any large decay by applying $|\theta|\partial_{|\theta|}$ many many times.

Example 3.5. *Note that in Example 3.1, the amplitude* $\frac{1-\chi(\xi)}{P(\xi)}$ *is in* $S^{-m}_{1,0}$.

Let

$$\|a\|_{K,\alpha,\beta} := \sup_{(x,\theta)\in K\times\mathbb{R}^N} \langle\theta\rangle^{-m+\rho|\beta|-\delta|\alpha|} |\partial_x^\alpha \partial_\theta^\beta a|,$$

then $a \in S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ if and only if for all $K \Subset X$, all multiindexes α, β, $\|a\|_{K,\alpha,\beta} < \infty$. And $S^m_{\rho,\delta}(X\times\mathbb{R}^n)$ is a Frechet space with the seminorms $\|a\|_{K,\alpha,\beta}$. A countable family of seminorms defining the topology is given by the $\|a\|_{K_j,\alpha,\beta}$, $j = 1, 2, \cdots, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^N$, where K_j is an increasing sequence of compact subsets of X such that $X = \cup_j K_j$.

Definition 3.6 (Topology of $S^m_{\rho,\delta}$ as a Frechet space)**.** $S^m_{\rho,\delta}$ *is metrizable with respect to the metric*

$$d(f,g) = \sum_{j=1}^\infty 2^{-j} \frac{\|f-g\|_j}{1+\|f-g\|_j}.$$

With respect to this topology, we say

$$a_j \to a \iff \|a_j - a\|_{K,\alpha,\beta} \to 0, \forall K, \alpha, \beta,$$

and the space is complete, that is, $S^m_{\rho,\delta}$ *is a Frechet space.*

The proof of the completeness is analogous to the proof that C^k functions form a Frechet space and the proof that Schwartz functions form a Frechet space.

Remark 3.7 (Why are these guys called symbols?). Suppose we have a general differential operator of order m

$$P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha.$$

If we apply it to some $u \in \mathscr{S}$, then

$$P(x,D)u = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha u = \frac{1}{(2\pi)^n} \iint \sum_{|\alpha|\le m} a_\alpha(x) \xi^\alpha e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi,$$

which makes perfect sense since we first take the Fourier transform of u and then multiply by ξ^α then take the inverse Fourier transform. On the other hand, you can think of this as an oscillatory integral only in ξ

$$\int \sum_{|\alpha|\le m} a_\alpha(x) \xi^\alpha e^{ix\cdot\xi}\, d\xi$$

like the delta function. Let $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ and this is called the symbol of $P(x,D)$ defined for differential operators. Obviously, $p \in S^m_{1,0}(X\times\mathbb{R}^n)$ since it is a polynomial.

Theorem 3.8. *Here are two basic properties.*
(1) Let $m \le m', \rho \ge \rho', \delta \le \delta'$*, then* $S^m_{\rho,\delta} \subset S^{m'}_{\rho',\delta'}$.
(2) $\partial_x^\alpha \partial_\theta^\beta$ *is continuous from* $S^m_{\rho,\delta}$ *to* $S^{m-\rho|\beta|+\delta|\alpha|}_{\rho,\delta}$.

Definition 3.9. *The residue space is defined as*
$S^{-\infty}(X\times\mathbb{R}^N) = \{a \in C^\infty(X\times\mathbb{R}^N) : \forall K \Subset X, \forall N, \exists C, s.t. \forall x \in K, |\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C\langle\theta\rangle^{-N}\}$.

Moreover, note the fact for the residue space $S^{-\infty}(X \times \mathbb{R}^N)$: Fix $(\rho,\delta) \in [0,1]^2$, then $S^{-\infty}(X \times \mathbb{R}^N) = \cap_m S^m_{\rho,\delta}(X \times \mathbb{R}^N)$.

Notation 3.10. *We will abbreviate* $S^m_{1,0}$ *as* S^m *since it is the most useful one.*

Now we present an example, a more general case compared to the amplitude function in Example 3.1.

Example 3.11. *Let* $a \in C^\infty(X \times \mathbb{R}^N)$ *and* $a(x,\lambda\theta) = \lambda^m a(x,\theta)$ *for* $|\theta| \ge 1, \lambda \ge 1$*, that is, a is homogeneous for* θ *large enough. Then* $a \in S^m_{1,0}$.

The proof is easy. Since the homogeneity gives us
$$\lambda^{|\beta|}\partial_\theta^\beta a(x,\lambda\theta) = \partial_\theta^\beta\left(a(x,\lambda\theta)\right) = \partial_\theta^\beta\left(\lambda^m a(x,\theta)\right) = \lambda^m \partial_\theta^\beta a(x,\theta),$$
which implies that
$$\partial_\theta^\beta a(x,\lambda\theta) = \lambda^{m-|\beta|}\partial_\theta^\beta a(x,\theta)$$
Hence, for all $\theta \in \mathbb{S}^{N-1}$, $\lambda \ge 1$,
$$\left|\partial_\theta^\beta a(x,\lambda\theta)\right| \le C(K,\beta)\langle\lambda\rangle^{m-|\beta|}.$$
that is, for all β, K,
$$|\partial_\theta^\beta a(x,\theta)| \le C_1(K,\beta)\langle\theta\rangle^{m-|\beta|}$$
holds for all $x \in K, |\theta| \ge 1$*. And for* $|\theta| \le 1$, $x \in K$*, there exists some* $C_2 > 0$ *such that* $|\partial_\theta^\beta a(x,\theta)| \le C_2(K,\beta)$*. Since* $\langle\theta\rangle^{m-|\beta|}$ *is between* 1 *and* $2^{m-|\beta|}$*, there exists*
$$C_3(K,\beta) = \max\{C_1(K,\beta), \min\{C_2(K,\beta), C_2(K,\beta)2^{m-|\beta|}\}\}$$
such that
$$|\partial_\theta^\beta a(x,\theta)| \le C_3(K,\beta)\langle\theta\rangle^{m-|\beta|}$$
for all $x \in K, \theta \in \mathbb{R}^n$*. Analogously, for all* α, β, K*, we have*
$$|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)| \le C(K,\alpha,\beta)\langle\theta\rangle^{m-|\beta|}$$
for all $x \in K, \theta \in \mathbb{R}^n$.

Example 3.12. *Let* $f \in C^\infty(X \times \mathbb{R}^N; [0,\infty))$ *such that* $f(x,\lambda\theta) = \lambda f(x,\theta)$ *for* $\lambda > 0$*, that is,* f *is positively homogeneous of degree* 1*. Set* $a(x,\theta) = e^{-f(x,\theta)}$*, then* $0 \le a \le 1$ *and* $a \in C^\infty$*. We claim that*
$$\partial_x^\alpha\partial_\theta^\beta(e^{-f}) = \sum_{|\widetilde{\alpha}|\le|\alpha|, |\widetilde{\beta}|\le|\beta|} a^{\alpha,\beta}_{\widetilde{\alpha},\widetilde{\beta}}(x,\theta)(\partial_x f)^{\widetilde{\alpha}}(\partial_\theta f)^{\widetilde{\beta}} e^{-f}, \tag{3.1}$$
where $a^{\alpha,\beta}_{\widetilde{\alpha},\widetilde{\beta}} \in S^{\frac{|\alpha|-|\widetilde{\alpha}|}{2} - \frac{|\beta|-|\widetilde{\beta}|}{2}}$ *and* $(\partial_x f)^\gamma = (\partial_{x_1} f)^{\gamma_1}(\partial_{x_2} f)^{\gamma_2}\cdots(\partial_{x_n} f)^{\gamma_n}$*. This claim can be proved by induction. Note that we absorb terms like* $\partial_x^{\widetilde{\alpha}}\partial_\theta^{\widetilde{\beta}} f$ *in* $a^{\alpha,\beta}_{\widetilde{\alpha},\widetilde{\beta}}(x,\theta)$ *and only keep the*

worst terms like $(\partial_x f)^{\widetilde{\alpha}}$, $(\partial_\theta f)^{\widetilde{\beta}}$ *in* (3.1)*. We only write out for the case* $|\alpha| = |\beta| = 1$ *to give a sense of what to prove :*

$$\partial_x^\alpha \partial_\theta^\beta (e^{-f}) = (-\partial_x^\alpha \partial_\theta^\beta f + \partial_\theta^\beta f \partial_x^\alpha f) e^{-f},$$

where $\partial_x^\alpha \partial_\theta^\beta f \in S^0$ *and* $1 \in S^0$*.*

Now we apply Landau's inequality which is proved in Lemma 3.13, then

$$|\partial_x f| + |\partial_\theta f| \leq C f^{\frac{1}{2}}$$

for all $x \in K$, $1 \leq |\theta| \leq 2$*. Thanks to the homogeneity, for* $\lambda > 0$,

$$\lambda^{-1} |\partial_x f(x, \lambda\theta)| + |\partial_\xi f(x, \lambda\theta)| \leq C \lambda^{-\frac{1}{2}} f(x, \theta)^{\frac{1}{2}}.$$

Take $\lambda = |\widetilde{\theta}|$, $\widetilde{\theta} = \lambda\theta$, $|\theta| = 1$*. Hence,*

$$|\theta|^{-\frac{1}{2}} |\partial_x f(x, \theta)| + |\theta|^{\frac{1}{2}} |\partial_\theta f(x, \theta)| \leq C f(x, \theta)^{\frac{1}{2}}.$$

Thus,

$$\left| (\partial_x f)^{\widetilde{\alpha}} (\partial_\theta f)^{\widetilde{\beta}} e^{-f} \right| \leq |\theta|^{\frac{|\widetilde{\alpha}|}{2}} |\theta|^{-\frac{|\widetilde{\beta}|}{2}} f^{\frac{|\widetilde{\alpha}| + |\widetilde{\beta}|}{2}} e^{-f} \leq C_{|\widetilde{\alpha}|, |\widetilde{\beta}|, K} \langle \theta \rangle^{\frac{|\widetilde{\alpha}| - |\widetilde{\beta}|}{2}}.$$

Combined with (3.1)*, here we get*

$$|\partial_x^\alpha \partial_\theta^\beta (e^{-f})| \leq C \langle \theta \rangle^{\frac{|\alpha| - |\beta|}{2}},$$

that is, $a(x, \theta) \in S^0_{\frac{1}{2}, \frac{1}{2}}(X \times \mathbb{R}^n)$*.*

Lemma 3.13 (Landau's inequality)**.** *Let* $g \in C^2(U)$, $g \geq 0$*, where* U *is an open set. Then for all compact sets* $K \Subset U$*, there exists* $C > 0$ *such that* $|\nabla g(x)| \leq C\sqrt{g(x)}$ *for all* $x \in K$*.*

Proof. The trick is that we use the Taylor formula here. We have

$$0 \leq g(x + y) = g(x) + \nabla g(x) \cdot y + O(|y|^2),$$

that is,

$$-\nabla g(x) \cdot y \leq g(x) + O(|y|^2).$$

Let $y = -\varepsilon \nabla g(x)$, where ε is small enough such that $x + y \in U$ for all $x \in K$. Furthermore, we choose ε small enough such that $O(\varepsilon^2 |\nabla g(x)|^2) \leq \frac{1}{2}\varepsilon |\nabla g(x)|^2$, then

$$\varepsilon |\nabla g(x)|^2 \leq g(x) + O(\varepsilon^2 |\nabla g(x)|^2)$$

implies $\frac{1}{2}\varepsilon |\nabla g(x)|^2 \leq g(x)$. Set $C = \frac{2}{\varepsilon}$, then the desired inequality follows. □

Remark 3.14. Take $g(x) = x^2$ in one dimension and you can see this equality is sharp.

Before we state the next theorem, we need another interesting lemma.

Lemma 3.15 (An interesting interpolation lemma)**.** *Suppose* $f \in C^2([-\varepsilon, \varepsilon])$*, then*

$$|f'(0)| \leq 2\|f\|_\infty^{\frac{1}{2}} \|f''\|_\infty^{\frac{1}{2}} + \frac{4}{\varepsilon} \|f\|_\infty,$$

where $\|g\|_\infty = \sup_{|x| \leq \varepsilon} |g(x)|$*.*

Proof. We write

$$f(x) = f(0) + xf'(0) + x^2 \int_0^1 (1-t) f''(tx)\, dt,$$

then

$$|f'(0)| \leq \frac{2}{x}\|f\|_\infty + \frac{x}{2}\|f''\|_\infty.$$

Note that the right hand side depends on x while the left hand side does not. We minimize this with respect to x, where the minimizer is $x = \min\left(2\frac{\|f\|_\infty^{\frac{1}{2}}}{\|f''\|_\infty^{\frac{1}{2}}}, \varepsilon\right)$. When $\varepsilon > 2\frac{\|f\|_\infty^{\frac{1}{2}}}{\|f''\|_\infty^{\frac{1}{2}}}$, $|f'(0)| \leq \frac{2}{\varepsilon}\|f\|_\infty + \frac{\varepsilon}{2}\|f''\|_\infty \leq \frac{4}{\varepsilon}\|f\|_\infty$. Hence, combined with the two cases,

$$|f'(0)| \leq 2\|f\|_\infty^{\frac{1}{2}}\|f''\|_\infty^{\frac{1}{2}} + \frac{4}{\varepsilon}\|f\|_\infty.$$

□

Remark 3.16. Note that we need the extra term $\|f\|_{L^\infty}$ on the right hand side since for linear functions, the second derivatives are zero.

Theorem 3.17. *Suppose $\{a_j\}$ is bounded in $S^m_{\rho,\delta}$ and $a_j(x,\theta) \to a(x,\theta)$ for all $(x,\theta) \in X \times \mathbb{R}^N$ pointwisely. Then $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and $a_j \to a$ in $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$ for all $m' > m$.*

Proof. Take $z = (x,\theta) \in X \times \mathbb{R}^N$ and $1 \leq l \leq n + N$. With a slight abuse of notation, we denote $\|\cdot\|_\infty$ be the supremum with respect to a small interval near z in the l-th slot $[z - \varepsilon e_l, z + \varepsilon e_l]$.

Since $\{a_j\}$ is bounded in $S^m_{\rho,\delta}$, $\{\|\partial_{z_l} a_j - \partial_{z_l} a_k\|_\infty\}_{j,k}$ and $\{\|\partial^2_{z_l} a_j - \partial^2_{z_l} a_k\|_\infty\}_{j,k}$ are bounded. Moreover, we apply Arzela-Ascoli theorem, then we get the uniform convergence by passing to a subsequence (also denoted by $\{a_j\}$) if necessary, that is, $\|a_j - a_k\|_\infty \to 0$. Hence, $\{a_j\}_j$ is a Cauchy sequence in $C(X \times \mathbb{R}^N)$.

Furthermore, Lemma 3.15 tells us

$$|\partial_{z_l} a_j(z) - \partial_{z_l} a_k(z)| \leq 2\|a_j - a_k\|_\infty \|\partial^2_{z_l} a_j - \partial^2_{z_l} a_k\|_\infty + \frac{4}{\varepsilon}\|a_j - a_k\|_\infty \to 0.$$

By the arbitrariness of $z \in X \times \mathbb{R}^N$, we know that $\partial_{z_l} a_j(x,\theta)$ converges for all $(x,\theta) \in X \times \mathbb{R}^N$, that is, $\partial a_j(x,\theta)$ converges pointwisely. Repeating the preceding argument, we will get a subsequence by Arzela-Ascoli such that $\|\partial a_j - \partial a_k\|_\infty \to 0$.

This allows us to obtain by induction that $\{a_j\}_j$ is a Cauchy sequence in $C^k(X \times \mathbb{R}^N)$ for all $k \in \mathbb{N}$ and hence for $k = \infty$.

Consequently, $a \in C^\infty(X \times \mathbb{R}^N)$ and $a_j \to a$ in $C^\infty(X \times \mathbb{R}^N)$. Then thanks to the uniform bound of $\{a_j\}_j$ in $S^m_{\rho,\delta}$ we know $a \in S^m_{\rho,\delta}$.

In order to prove the convergence in $S^{m'}_{\rho,\delta}$, we let $K \Subset X$ and consider $(x,\theta) \in K \times \mathbb{R}^N$. Let

$$k_j(x,\theta) := \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{\langle\theta\rangle^{m' - \rho|\beta| + \delta|\alpha|}} = \frac{1}{\langle\theta\rangle^{m'-m}} \cdot \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{\langle\theta\rangle^{m - \rho|\beta| + \delta|\alpha|}}.$$

We know that $\frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{\langle\theta\rangle^{m - \rho|\beta| + \delta|\alpha|}}$ is uniformly bounded and goes to 0 on compact sets. Moreover, $\frac{1}{\langle\theta\rangle^{m'-m}}$ goes to 0 as $|\theta| \to \infty$. So we can estimate by two parts.

For all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $|k_j| < \varepsilon$ if $|\theta| \geq R_\varepsilon$. On the other hand, for $|\theta| \leq R_\varepsilon, x \in K$, there exists J_ε, such that $|k_j| < \varepsilon$ if $j > J_\varepsilon$. Hence, $|k_j| < \varepsilon$ if $j > J_\varepsilon$ and $x \in K$, which implies

$$a_j \to a \text{ in } S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N).$$

□

Now here is an example showing that convergence in $S^m_{\rho,\delta}$ is always true. In other words, the preceding result is in some sense sharp.

Example 3.18. *Let $a = 1$, $a_j = \chi(\frac{\theta}{j})$, where $\chi \equiv 1$ near 0. Then $a_j(\theta) \to a(\theta)$ for all θ. We know that a is smooth and in S^0. Note that $\|a_j - a\|_\infty = 1$, so a_j does not converge to a in S^0. However, for any $\delta > 0$, $\|\langle\theta\rangle^{-\delta}(a_j - a)\|_\infty \to 0$ as $j \to \infty$.*

Theorem 3.19. *For every $m' > m$, $S^{-\infty}(X\times\mathbb{R}^N)$ is dense in $S^m(X\times\mathbb{R}^N)$ in the topology of $S^{m'}(X \times \mathbb{R}^N)$.*

Proof. Set $a \in S^m(X \times \mathbb{R}^N)$. Let $\chi_j(\theta) = \chi(\frac{\theta}{j})$ where $\chi \equiv 1$ near 0 and $\chi \in C_c^\infty$. Then for all $|\alpha| > 0$,

$$\partial_\theta^\alpha \chi_j(\theta) = j^{-|\alpha|}\chi^{(\alpha)}(\frac{\theta}{j}).$$

Note that $\chi^{(\alpha)}(\frac{\theta}{j})$ is supported in $j \leq |\theta| \leq 2j$ without loss of generality, which implies that the right hand side is $O(\langle\theta\rangle^{-|\alpha|})$ and the bound is independent of j. This tells us $\chi_j \in S^0_{1,0}$. Then $a_j := \chi_j a \in S^m_{\rho,\delta}$ and $\{a_j\}$ is a uniformly bounded sequence in $S^m_{\rho,\delta}$. Moreover, a_j is compactly supported in θ, so $a_j \in S^{-\infty}$. Furthermore, by Theorem 3.17, we know that $a_j \to a \in S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$, which completes the proof. □

Remark 3.20. This theorem is the same type of thing we present before. Note that C_c^∞ or rapidly decay functions are not dense in L^∞ in the topology of L^∞. However, they are dense if you put some weight like $\langle\theta\rangle^{-\delta}$ as in Example 3.18, that is, $\mathscr{S}$ is dense in continuous functions in the topology of $\langle x\rangle^\delta L^\infty$ for all $\delta > 0$.

Now we make some philosophical comments. Note that $\sum_{j=0}^\infty a_j h^j$ could converge provided $|a_j| \leq C^j$, where $a_j \in \mathbb{C}, 0 < h \ll 1$. When the sum does not converge, we say

$$a \sim \sum_{j=0}^\infty a_j h^j$$

if for all N, there exists C_N such that $|a - \sum_{j=0}^{N-1} a_j h^j| \leq C_N h^N$.

We have seen this thing in the Taylor series. The Taylor series $\sum_{n=0}^\infty \frac{u^{(n)}(x)}{n!}y^n$ may not converge, but we have

$$u(x+y) = \sum_{n=0}^{N-1} \frac{u^{(n)}(x)}{n!}y^n + \frac{y^N}{(N-1)!}\int_0^1 u^{(N)}(x+sy)(1-s)^{N-1}\,ds,$$

where the remainder is $O(|y|^N)$ so that $u(x+y) \sim \sum_{n=0}^\infty \frac{u^{(n)}(x)}{n!}y^n$.

Theorem 3.21. *Suppose $a_j \in S^{m_j}_{\rho,\delta}$ with $m_j \to -\infty$ and $m_0 \geq m_1 \geq m_2 > \cdots$. Then there exists $a \in S^{m_0}_{\rho,\delta}$, for all k,*

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{\rho,\delta}(X \times \mathbb{R}^N). \tag{3.2}$$

And a is unique in the sense that if (3.2) holds for another $\widetilde{a}$, then $\widetilde{a} - a \in S^{-\infty}$. We will denote this by

$$a \sim \sum_{j=0}^{\infty} a_j.$$

Proof. Let $\|\cdot\|_{k,l}$ be a sequence of seminorms defining for $S^{m_k}_{\rho,\delta}$. Without loss of generality, we assume $m_0 > m_1 > m_2 > \cdots$. By density (Theorem 3.19), for all j, there exists $b_j \in S^{-\infty}$ such that

$$\|a_j - b_j\|_{\nu,\mu} \leq 2^{-j}, \quad 0 \leq \nu, \mu \leq j-1.$$

(This is another sort of diagonal argument.) This is true because $S^{-\infty}$ is dense in $S^{m_j}_{\rho,\delta}$ in the topology of $S^{m_\nu}_{\rho,\delta}$ for $\nu \leq j-1$. Hence, for all k, l, we have

$$\|\sum_{j>k}(a_j - b_j)\|_{k,l} \leq \sum_{j>k} \|a_j - b_j\|_{k,l} \leq \sum_{j\leq l} \|a_j - b_j\|_{k,l} + \sum_{j>l} 2^{-j} < \infty.$$

Thus, $\sum_{j\geq k}(a_j - b_j)$ converges in $S^{m_k}_{\rho,\delta}$ for all k.

Now we set $a := \sum_{j=0}^{\infty}(a_j - b_j) \in S^{m_0}_{\rho,\delta}$. And we check by calculating

$$a - \sum_{j<k} a_j = -\sum_{j<k} b_j + \sum_{j=k}^{\infty}(a_j - b_j) \in S^{-\infty} + S^{m_k}_{\rho,\delta} \subset S^{m_k}_{\rho,\delta}.$$

Note that $a - \widetilde{a} \in S^{m_k}_{\rho,\delta}$ for all k. But $m_k \to -\infty$, we know $a - \widetilde{a} \in S^{-\infty}$. □

3.2. Phase functions and Oscillatory integrals.

We denote $\dot{\mathbb{R}}^N = \mathbb{R}^N \backslash \{0\}$.

Definition 3.22 (Non-degenerate phase function)**.** *A function $\varphi = \varphi(x, \theta)$ is called a non-degenerate phase function if for all $(x, \theta) \in X \times \dot{\mathbb{R}}^N$*

(1) $\varphi \in C^\infty(X \times \dot{\mathbb{R}}^N)$,
(2) $\varphi(x, \lambda\theta) = \lambda\varphi(x, \theta)$ for all $\lambda > 0$,
(3) Im $\varphi(x, \theta) \geq 0$,
(4) $d\varphi \neq 0$, where the differential is defined by $d\varphi = \sum_j \partial_{\theta_j}\varphi d\theta_j + \sum_j \partial_{x_j}\varphi dx_j$. And here $d\varphi \neq 0$ means that $(\partial_{\theta_1}\varphi, \cdots, \partial_{\theta_N}\varphi, \partial_{x_1}\varphi, \cdots, \partial_{x_n}\varphi) \neq 0$.

Lemma 3.23. *Suppose $m + k < -N$ and φ is a non-degenerate phase function, then*

$$a \mapsto I(a, \varphi) := \int_{\mathbb{R}^N} a(x, \theta) e^{i\varphi(x,\theta)}\, d\theta$$

defines a continuous map from $S^m_{\rho,\delta}(X \times \mathbb{R}^N) \to C^k(X)$.

Proof. Since $I(a,\varphi) = \int_{\mathbb{R}^N} a(x,\theta)e^{i\varphi(x,\theta)}\,d\theta$, if $m < -N - \varepsilon$, then $|a| \le \langle\theta\rangle^{-N-\varepsilon}$ and hence the integral converges.

If we start differentiating with respect to x, then we know for $|\alpha| = 1$,

$$\partial_x^\alpha I(a,\varphi) = \int_{\mathbb{R}^N} \left(\partial_x^\alpha a(x,\theta) + i\partial_x^\alpha\varphi(x,\theta)a(x,\theta)\right)e^{i\varphi(x,\theta)}\,d\theta.$$

Note that $\partial_x^\alpha a(x,\theta) \in S^{m+\delta|\alpha|}_{\rho,\delta}$, $\partial_x^\alpha\varphi(x,\theta) \in S^1 \subset S^1_{\rho,\delta}$ (Example 3.11) and $e^{i\varphi(x,\theta)}$ is bounded, we know that $\partial_x^\alpha I(a,\varphi) \in C(X)$ when $m < -N - 1 - \varepsilon$.

We iterate this process and we see if $m + k < -N$, we have $I(a,\varphi) \in C^k(X)$. ☐

Corollary 3.24. *If $a \in S^{-\infty}$, then $I(a,\varphi) \in C^\infty(X)$.*

In the following theorem, we need to impose some restriction on ρ, δ.

Theorem 3.25. *Assume φ is a non-degenerate phase function. Let $0 < \rho \le 1$ and $0 \le \delta < 1$. There exists a unique way of defining $I(a,\varphi) \in \mathcal{D}'(X)$ for $a \in \cup_m S^m_{\rho,\delta}$ such that for $a \in S^m_{\rho,\delta}$, $m < -N$, $I(a,\varphi)$ is given by $I(a,\varphi) = \int a(x,\theta)e^{i\varphi(x,\theta)}\,d\theta$ and for all $m \in \mathbb{R}$,*

$$S^m_{\rho,\delta} \to \mathcal{D}'(X), \quad a \mapsto I(a,\varphi)$$

is a continuous map.

More precisely, if $k \in \mathbb{N}$ and $m - k\min(\rho, 1-\delta) < -N$, then the map

$$S^m_{\rho,\delta} \ni a \mapsto I(a,\varphi) \in \mathcal{D}'^{(k)}(X)$$

is continuous from symbols to the distributions of order k.

Proof. **Uniqueness:** For all $m \ge -N$, choose $m' > m$, then $S^{-\infty}$ is dense in $S^m_{\rho,\delta}$ in the topology of $S^{m'}_{\rho,\delta}$ by Theorem 3.19. Since $S^{m'}_{\rho,\delta} \to \mathcal{D}'(X)$ is continuous and $S^m_{\rho,\delta} \subset S^{m'}_{\rho,\delta}$, $I(a,\varphi)$ is uniquely defined on $S^m_{\rho,\delta}$ by density.

Existence: The main idea of the proof is to use $d\varphi \ne 0$ to find differential operator L such that ${}^tL(e^{i\varphi}) = e^{i\varphi}$. And then for $a \in S^{-\infty}$, $I(a,\varphi)v = \int L^k(av)e^{i\varphi}\,d\theta\,dx$, where $v \in C_c^\infty$. Because $L^k a$ will improve the decay of a, we will be able to define the distribution.

The proof is based on the following lemma.

Lemma 3.26. *Suppose φ is a non-degenerate phase function. Then there exists $a_j \in S^0_{1,0}, b_l \in S^{-1}_{1,0}, c \in S^{-1}_{1,0}$ such that the differential operator L is defined by*

$$L := \sum_{j=1}^N a_j(x,\theta)\partial_{\theta_j} + \sum_{l=1}^N b_l(x,\theta)\partial_{x_l} + c(x,\theta)$$

such that ${}^tL(e^{i\varphi}) = e^{i\varphi}$. And the transpose ${}^tL$ is the formal operator satisfying $Lv(u) = v({}^tLu)$.

Proof. Choose $\chi \in C_c^\infty(\mathbb{R}^N)$ and $\chi \equiv 1$ near 0. Let

$$\Phi = \sum\left|\frac{\partial\varphi}{\partial x_j}\right|^2 + |\theta|^2\sum\left|\frac{\partial\varphi}{\partial\theta_j}\right|^2 = \sum\overline{\frac{\partial\varphi}{\partial x_j}}\frac{\partial\varphi}{\partial x_j} + |\theta|^2\sum\overline{\frac{\partial\varphi}{\partial\theta_j}}\frac{\partial\varphi}{\partial\theta_j}.$$

Note that $\Phi \in C^\infty(X \times \dot{\mathbb{R}}^N)$ and Φ is homogeneous of degree 2. Moreover, $\Phi \ne 0$ if $\theta \ne 0$ since φ is non-degenerate.

We define

$$ {}^tL = \frac{1-\chi(\theta)}{i\Phi}\left(|\theta|^2 \sum \overline{\frac{\partial\varphi}{\partial\theta_j}}\partial_{\theta_j} + \sum \overline{\frac{\partial\varphi}{\partial x_j}}\partial_{x_j}\right) + \chi(\theta) = \sum a'_j \partial_{\theta_j} + \sum b'_j \partial_{x_j} + c', $$

where

$$ a'_j = \frac{1-\chi(\theta)}{i\Phi}|\theta|^2 \frac{\partial\varphi}{\partial\theta_j} \in S^0, \quad b'_j = \frac{1-\chi(\theta)}{i\Phi}\frac{\partial\varphi}{\partial x_j} \in S^{-1}, \quad c' = \chi(\theta) \in S^{-\infty} \subset S^{-1} $$

since $1-\chi(\theta) \in S^0, \Phi \in S^2, \frac{\partial\varphi}{\partial\theta_j} \in S^0, \frac{\partial\varphi}{\partial x_j} \in S^1$. And if we apply $|\theta|^2 \sum \overline{\frac{\partial\varphi}{\partial\theta_j}}\partial_{\theta_j} + \sum \overline{\frac{\partial\varphi}{\partial x_j}}\partial_{x_j}$ to φ, we get $i\Phi$. Thus, we have

$$ {}^tL(e^{i\varphi}) = e^{i\varphi}. $$

Furthermore, ${}^t(b\partial_{x_j}) = -b\partial_{x_j} - (\partial_{x_j} b)$ and ${}^t(a\partial_{\theta_j}) = -a\partial_{\theta_j} - (\partial_{\theta_j} a)$,

$$ L = {}^t({}^tL) = -\sum a'_j\partial_{\theta_j} - \sum b'_j \partial_{x_j} + c' - \sum \partial_{x_j} b'_j - \sum \partial_{\theta_j} a'_j. $$

Hence, $a_j = -a'_j \in S^0, b_j = -b'_j \in S^{-1}, c = c' - \sum \partial_{x_j} b'_j - \sum \partial_{\theta_j} a'_j \in S^{-1}$, which completes the proof. □

Now we go back to the proof of the existence in Theorem 3.25.

Step 1: We first consider the case $a \in S^{-\infty}$. Now we know $I(a,\varphi) \in C^\infty(X)$ by Corollary 3.24. For $u \in C_c^\infty(X)$, we consider the distributional pairing

$$ \begin{aligned} \langle I(a,\varphi), u\rangle = \int_X \int_{\mathbb{R}^N} a(x,\theta)u(x)e^{i\varphi(x,\theta)}\,d\theta\,dx &= \int_X \int_{\mathbb{R}^N} a(x,\theta)u(x)({}^tL)^k e^{i\varphi(x,\theta)}\,d\theta\,dx \\ &= \int_X \int_{\mathbb{R}^N} L^k\left(a(x,\theta)u(x)\right)e^{i\varphi(x,\theta)}\,d\theta\,dx. \end{aligned} $$

Step 2: Now suppose $a \in S^m_{\rho,\delta}$ for $\rho > 0, \delta < 1$. Since $a_j\partial_{\theta_j}(au) \in S^{m-\rho}_{\rho,\delta}$, $b_j\partial_{x_j}(au) \in S^{m-(1-\delta)}_{\rho,\delta}$ and $cau \in S^{-1}_{\rho,\delta}$, we have $L^k(au) \in S^{m-k\min(\rho,1-\delta)}_{\rho,\delta}$. Since $\rho, 1-\delta > 0$, we gain decays. And the map

$$ L^k : S^m_{\rho,\delta} \times C_0^\infty(X) \to S^{m-k\min(\rho,1-\delta)}_{\rho,\delta}, \quad (a(x,\theta), u(x)) \mapsto L^k(au) $$

is continuous since we can conclude from a more precise calculation as above that for every compact set $K \subset \Omega$,

$$ \sup_{K\times\mathbb{R}^N} |L^k(au)|\langle\theta\rangle^{-m+k\min(\rho,1-\delta)} \le \sum_{|\alpha|+|\beta|\le k} \|a\|_{K,\alpha,\beta} \sum_{|\alpha|\le k} \sup_K |\partial^\alpha u(x)|, \tag{3.3} $$

where $\|\cdot\|_{K,\alpha,\beta}$ are seminorms for $S^m_{\rho,\delta}$. (Similar estimates hold if we put higher seminorms for $S^{m-k\min(\rho,1-\delta)}_{\rho,\delta}$ in the left hand side.)

Step 3: For $a \in S^m_{\rho,\delta}$, we choose k such that $m - k\min(\rho, 1-\delta) < -N$, then we define a distribution $I_k(a,\varphi)$ as

$$ \langle I_k(a,\varphi), u\rangle = \iint L^k(au)e^{i\varphi}\,d\theta\,dx. $$

The integral on the right hand side converges absolutely since $L^k(au) \in S^{m-k\min(\rho,1-\delta)}$ and it is compactly supported in x. And it defines a distribution since the seminorm estimates gives (3.3)

$$|\langle I_k(a,\varphi), u\rangle| \le C_{a,k,\mathrm{supp}u} \sum_{|\alpha|\le k} |\partial^\alpha u|,$$

and more precisely, $I_k(a,\varphi) \in \mathcal{D}'^{(k)}(X)$ and the map

$$S^m_{\rho,\delta}(X\times\mathbb{R}^N) \to \mathcal{D}'^{(k)}(X), \quad a \mapsto I_k(a,\varphi)$$

is continuous. And for $a \in S^{-\infty}$, by integration by parts in Step 1, $I_k(a,\varphi) = I(a,\varphi)$.

Step 4: To define $I(a,\varphi)$ for $a \in S^m_{\rho,\delta}$, we need to show that if $m - k'\min(\rho, 1-\delta) < -N$, $I_k(a,\varphi) = I_{k'}(a,\varphi)$. For $a \in S^m_{\rho,\delta}$, we choose $a_j \in S^{-\infty}$ such that $a_j \to a$ in $S^{m'}_{\rho,\delta}$ for $m' > m$ and $m' - k'\min(\rho, 1-\delta) < -N$. Since $I_k(a_j,\varphi) = I_{k'}(a_j,\varphi)$, by continuity, we know $I_k(a,\varphi) = I_{k'}(a,\varphi)$. Hence, we can define $I(a,\varphi) = I_k(a,\varphi)$.

Step 5: In Step 4, we showed that the definition of I is well-defined, which is independent of k. The uniqueness tells us the definition is also independent of which L we choose as long as coefficients satisfy the desired properties. ☐

Remark 3.27. Set $\chi \in C_c^\infty$ such that $\chi \equiv 1$ near 0. Actually, we have

$$I(a,\varphi) = \lim_{\varepsilon\to 0+} \int \chi(\varepsilon\theta) a(x,\theta) e^{i\varphi(x,\theta)}\, d\theta, \tag{3.4}$$

where the limit above is taken in the sense of distribution. This fact is simply because $a_\varepsilon = a(x,\theta)\chi(\varepsilon\theta) \in S^{-\infty}$ (see the proof of Theorem 3.19) and $a_\varepsilon \to a$ in $S^{m'}_{\rho,\delta}$ for all $S^m_{\rho,\delta}$ by Theorem 3.17. Hence, we have the limit due to Theorem 3.25.

For later use, if $\widetilde{L} = \sum_j a_j(x,\theta)\partial_{\theta_j} + \sum_j b_j(x,\theta)\partial_{x_j} + c(x,\theta)$ with $a_j \in S^0_{\rho,\delta}$, $b_j, c \in S^{-1}_{\rho,\delta}$, then for $u \in C_0^\infty(X)$, the integration by parts formula

$$\iint {}^t\widetilde{L}(e^{i\varphi}) au = \iint e^{i\varphi}\widetilde{L}(au) \tag{3.5}$$

holds with the understanding that these are oscillatory integrals, that is, even if those are formally expressions (only makes sense as oscillatory integrals), we can integrate by parts if the condition above is satisfied. The proof just follows from the proof of Theorem 3.25 as mentioned in Step 5. The proof does not rely on the specific form of L we choose and in the preceding lemma, we just give an explicit (non-unique) construction for such an L. In short, the proof idea is as follows. For $a \in S^{-\infty}$, (3.5) holds obviously. Thanks to the desired symbol spaces which a_j, b_j, c are in, we can prove the maps $a \mapsto \int e^{i\varphi}\widetilde{L}(a\cdot)\,\theta$ and $\int {}^t\widetilde{L}(e^{i\varphi})a\, d\theta$ are continuous map from $S^m_{\rho,\delta}$ to $\mathcal{D}'(X)$. Then by density, we know (3.5) holds for all $a \in S^m_{\rho,\delta}$.

Example 3.28. *We take* $\delta_0(x) = \int_{\mathbb{R}^n} e^{ix\cdot\theta}\, d\theta$. *Set* $V = \dfrac{(1-\chi(x))x\cdot\partial_\theta}{i|x|^2}$, *and note that* $\dfrac{(1-\chi(x))x}{|x|^2} \in S^0$, *which satisfies* (3.5). *Thus we can integrate by parts and get*

$$\int_{\mathbb{R}^n} e^{ix\cdot\theta}\, d\theta = 0$$

away from zero (when x is away from $\mathrm{supp}\chi$*).*

On the other hand, we can compute it using Remark 3.27. We choose χ *as in* (3.4)*. Then for* $u \in C_0^\infty(X)$*, we have*

$$\langle \int_{\mathbb{R}^n} e^{ix\cdot\theta}\, d\theta, u\rangle = \lim_{\varepsilon\to 0} \iint \chi(\varepsilon\theta)u(x)e^{ix\cdot\theta}\, d\theta\, dx = \lim_{\varepsilon\to 0}\int \chi(\varepsilon\theta)\widehat{u}(-\theta)\, d\theta = \int \widehat{u}(-\theta)\, d\theta = u(0).$$

Hence, when $\int_{\mathbb{R}^n} e^{ix\cdot\theta}\, d\theta$ *is meant as a oscillatory integral, it is indeed the delta function* δ_0*.*

Definition 3.29 (Critical set)**.** *If* φ *is a non-degenerate phase function, we call*

$$C_\varphi = \{(x,\theta) \in X \times \dot{\mathbb{R}}^N : d_\theta\varphi = 0\}$$

the critical set of φ*.*

Example 3.30. *For* $\delta_0(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\theta}\, d\theta$*, where* $\varphi = x\cdot\theta$*, we have*

$$C_\varphi = \{(0,\theta) : \theta \in \dot{\mathbb{R}}^n\}.$$

Definition 3.31. *We say* C *is a cone if for all* $x \in C$*,* $\lambda > 0$*, we have* $\lambda x \in C$*. For* $\theta_0 \in \mathbb{R}^N$*, we call the set of the form*

$$\left\{\theta \in \mathbb{R}^N : \left|\frac{\theta}{|\theta|} - \frac{\theta_0}{|\theta_0|}\right| < \varepsilon\right\}$$

for some $\varepsilon > 0$ *a conic neighborhood of* θ_0*. Note that a conic neighborhood is a cone.*

Lemma 3.32. *Suppose* $a \in S^m_{\rho,\delta}$*,* $\rho > 0, \delta < 1$ *and* $a \equiv 0$ *in a conic neighborhood of* C_φ*(a neighborhood which is conic in* θ*). Then* $I(a,\varphi) \in C^\infty(X)$*.*

Proof. First note that C_φ is a cone in θ thanks to homogeneity, then the statement makes perfect sense. Now we claim that there exists a differential operator L defined by

$$L = \sum a_j\partial_{\theta_j} + c, \quad a_j \in S^0,\ c \in S^{-1} \tag{3.6}$$

such that ${}^tL(e^{i\varphi}) = (1-b)e^{i\varphi}$ for $b \in S^0$ and $\operatorname{supp} b \cap \operatorname{supp} a = \varnothing$. That is, ${}^tL(e^{i\varphi}) = e^{i\varphi}$ holds on the support of a.

Step 1: We construct a function b homogeneous of degree 0 in θ for $|\theta| > 1$ such that $b \equiv 1$ in a conic neighborhood of C_φ when $|\theta| \geq 1$ and $b \equiv 0$ on $\operatorname{supp} a$.

Denote the conic neighborhood in the assumption of the lemma by

$$C_N = \left\{(x,\theta) : |x'-x| < \varepsilon, \left|\frac{\theta'}{|\theta'|} - \frac{\theta}{|\theta|}\right| < \varepsilon, \forall (x',\theta') \in C_\varphi\right\}.$$

We set $b(x,\theta) \equiv 1$ in

$$\left\{(x,\theta) : |\theta| \geq 1, |x'-x| < \frac{\varepsilon}{2}, \left|\frac{\theta'}{|\theta'|} - \frac{\theta}{|\theta|}\right| < \frac{\varepsilon}{2}, \forall (x',\theta') \in C_\varphi\right\}.$$

In other words, we set $b \equiv 1$ in a neighborhood of $X \times \mathbb{S}^{N-1}$ which is disjoint from $\operatorname{supp} a$ and extend it by homogeneity for $|\theta| \geq 1$. Then we let $b \equiv 0$ outside C_N and extend it by homogeneity in θ and make it smooth. At last, we cut it off near $|\theta| = 0$. Then we obtained the desired $b \in S^0$. (We do this since all the oscillatory nature happens at infinity. So we do not need to consider things near 0.)

Step 2: Let ${}^tL = (1-b(x,\theta))\left(\frac{1-\chi(\theta)}{i|\varphi_\theta|^2}\langle\varphi_\theta,\partial_\theta\rangle + \chi(\theta)\right)$, where we choose $\chi \equiv 1$ when $|\theta| \le 1$. Note that C_φ is the set where φ_θ vanishes and but $1-b$ near C_φ except for θ near 0. However, $1-\chi$ vanishes when θ near 0, so ${}^tL$ is perfectly well-defined.

Since $\varphi_\theta \in S^0$, ${}^tL = \sum a'_j\partial_{\theta_j} + c'$ with $a'_j \in S^0$ and $c' \in S^{-1}$. Then it follows from a similar computation as in the proof of Lemma 3.26 that $L = \sum a_j\partial_{\theta_j} + c$ with $a \in S^0$ and $c \in S^{-1}$, which proves the claim.

Step 3: Thus, we can integrate by parts as follows

$$\begin{aligned}\langle I(a,\varphi),u\rangle &= \iint e^{i\varphi(x,\theta)}a(x,\theta)u(x)\,dx\,d\theta = \iint ({}^tL)^k\left(e^{i\varphi(x,\theta)}\right)a(x,\theta)u(x)\,dx\,d\theta\\ &= \iint e^{i\varphi(x,\theta)}L^k\left(a(x,\theta)u(x)\right)\,dx\,d\theta = \iint e^{i\varphi(x,\theta)}L^k\left(a(x,\theta)\right)u(x)\,dx\,d\theta,\end{aligned}$$

where in the second equality we use the support property in the claim at the beginning. Hence, $I(a,\varphi) = I(L^ka,\varphi)$ for any k. But we know $L^ka \in S^{m-k\min(\rho,1-\delta)}$, where $m - k\min(\rho,1-\delta)$ can be sufficiently negative for k large. So $I(a,\varphi) \in C^l(X)$ for all l by Lemma 3.23. □

Corollary 3.33. *If a is supported away from C_φ, then $I(a,\varphi) \in C^\infty(X)$.*

Now we recall the definition of the singular support of a distribution.

Definition 3.34 (Singular support of a distribution)**.** *For $u \in \mathcal{D}'(X)$, the singular support of u is defined by*

$$\begin{aligned}\text{singsupp}u &= \complement\{x : \exists U = \text{ a neighborhood of } x, u|_U \in C^\infty(X)\}\\ &= \text{ the smallest closed subset } L \subset X \text{ such that } u|_{X\setminus L} \in C^\infty,\end{aligned} \tag{3.7}$$

where $\complement$ denotes the complement of the set and here we mean the restriction of a distribution $u|_U$ by only applying this distribution to a smooth function compactly supported in U.

Theorem 3.35. *Let $\pi : X \times \dot{\mathbb{R}}^N \to X$ such that $\pi(x,\theta) = x$. Then*

$$\text{singsupp}I(a,\varphi) \subset \pi(C_\varphi).$$

Proof. Suppose $x_0 \notin \pi(C_\varphi)$. Note that C_φ is closed, then $\pi(C_\varphi)$ is closed (since φ is homogeneous of degree 1 in θ, so we can consider $\theta \in S^1$ compact), thus there exists $\psi \in C_c^\infty(X)$ such that $\psi(x_0) = 1$ and $\text{supp}\psi \cap \pi(C_\varphi) = \varnothing$.

We observe the identity $\psi I(a,\varphi) = I(\psi a,\varphi)$. From the support property of ψ, $\text{supp}(\psi a) \cap C_\varphi = \varnothing$, we can apply Lemma 3.32 to obtain $I(\psi a,\varphi) \in C^\infty(X)$. Therefore, $x_0 \notin \text{singsupp}I(a,\varphi)$. □

Remark 3.36. The formula

$$\psi I(a,\varphi) = I(\psi a,\varphi)$$

we used in the previous proof can be verified by density. It holds for $a \in S^{-\infty}$ obviously. Then we know the map $a \mapsto \psi I(a,\varphi)$ and $a \mapsto I(\psi a,\varphi)$ are all continuous since we can use (3.3) to check.

Example 3.37. *Suppose $f \in C^\infty(X)$, $Imf \geq 0$ and $f = 0$ implies $df(x) \neq 0$. Set $N = 1$ in this example, then*

$$u(x) = \int_0^\infty e^{if(x)\tau}\,d\tau = \int_0^\infty \widetilde{\chi}(\tau)e^{if(x)\tau}\,d\tau + \int_{\mathbb{R}} (1-\widetilde{\chi}(\tau))e^{if(x)\tau}\,d\tau,$$

where $\widetilde{\chi} \in C^\infty$ such that $\widetilde{\chi}(\tau) \equiv 1$ when $\tau < 1$ and $\widetilde{\chi}(\tau) \equiv 0$ when $\tau > 2$.

Obviously, the first part $\int_0^\infty \widetilde{\chi}(\tau)e^{if(x)\tau}\,d\tau$ is smooth and

$$\int_0^\infty \widetilde{\chi}(\tau)e^{i(f(x)\tau+i\varepsilon\tau)}\,d\tau \to \int_0^\infty \widetilde{\chi}(\tau)e^{if(x)\tau}\,d\tau$$

as $\varepsilon \to 0$ by the dominated convergence theorem.

And the second part is $I(1-\widetilde{\chi}, f(x)\tau)$ where $1-\widetilde{\chi} \in S^0$. Note that $(1-\widetilde{\chi}(\tau))e^{-\varepsilon\tau} \to 1-\widetilde{\chi}(\tau)$ in S^0, then by density and continuity in Theorem 3.25 and combined with the limit result above, we have

$$u(x) = \int_0^\infty e^{if(x)\tau}\,d\tau = \lim_{\varepsilon\to 0}\int_0^\infty e^{i(f(x)\tau+i\varepsilon\tau)}\,d\tau,$$

where the limit is taken in $\mathcal{D}'(X)$. Hence,

$$u(x) = \lim_{\varepsilon\to 0}\frac{i}{f(x)+i\varepsilon} =: \frac{i}{f(x)+i0}.$$

A particular case of interest is when $f(x) = x$. From this formula and $C_\varphi = \{(x,\tau) : f(x) = 0\}$, it is obvious that

$$singsuppu \subset \pi(C_\varphi),$$

which verifies Theorem 3.35.

Remark 3.38. The condition

$$f = 0 \Rightarrow df(x) \neq 0$$

in the assumption implies that $\{f(x) = 0\}$ is a hypersurface, that is, we can write it as a function in $n-1$ variable. This follows from the implicit function theorem.

Example 3.39. *We consider*

$$(\partial_t^2 - \Delta)u = 0, \quad u|_{t=0} = f,\ \partial_t u|_{t=0} = 0,$$

where $f \in \mathscr{S}(\mathbb{R}^n)$.

We claim that

$$u(t,x) = \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\cos(t|\xi|)f(y)\,dy\,d\xi = \frac{1}{2}\frac{1}{(2\pi)^n}\iint\sum_{\pm} e^{i((x-y)\cdot\xi\pm|\xi|t)}f(y)\,dy\,d\xi$$

is a solution to the wave equation. We check this by computing

$$u(0,x) = \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}f(y)\,dy\,d\xi = f(x),$$

$$\partial_t u(0,x) = \frac{1}{2}\frac{1}{(2\pi)^n}\iint\sum_{\pm}\pm|\xi|e^{i(x-y)\cdot\xi}f(y)\,dy\,d\xi = 0.$$

And you check the wave equation is satisfied by direct calculation. So consequently, the solution to the wave equation is defined by the following oscillatory integral

$$U(t,x,y) := \frac{1}{2}\frac{1}{(2\pi)^n}\int \sum_{\pm} e^{i((x-y)\cdot\xi \pm |\xi| t)}\, d\xi \in \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^n \times \mathbb{R}^n)$$

since $\varphi_\pm(t,x,y,\xi) = (x-y)\cdot\xi \pm |\xi| t$ are homogeneous of degree 1 in ξ and non-degenerate. Now,

$$\pi\left(C_{\varphi_\pm}\right) = \pi\left\{(t,x,y,\theta) : x-y+\frac{\theta}{|\theta|}t = 0\right\} = \{(t,x,y) : |x-y| = t\},$$

which defines the cone which we have already seen before when solving the wave equation.

3.3. **Generalizations of Oscillatory integrals.** Let $\varphi \in C^\infty(X\times Y\times\mathbb{R}^N)$, $X\subset\mathbb{R}^n, Y\subset\mathbb{R}^m$, which is a non-degenerate phase function on $(X\times Y)\times\mathbb{R}^N$. Here $X\times Y$ is our "old" X. For $a\in S^m_{\rho,\delta}(X\times Y\times\mathbb{R}^N)$, $\rho>0$, $\delta<1$, we get a distribution

$$K(x,y) = \int a(x,y,\theta)e^{i\varphi(x,y,\theta)}\, d\theta \in \mathcal{D}'(X\times Y).$$

Then by Schwartz kernel theorem, K defines an continuous map

$$A : C_c^\infty(Y)\to\mathcal{D}'(X)$$

such that for all $u\in C_c^\infty(Y), v\in C_c^\infty(X)$,

$$\langle Au, v\rangle := \langle K, v\otimes u\rangle. \tag{3.8}$$

Formally, we write

$$Au(x) = \iint a(x,y,\theta)e^{i\varphi(x,y,\theta)}u(y)\, dy\, d\theta, \tag{3.9}$$

which is a convenient way of writing things out. However, it always denotes in the sense of distributional pairing (3.8).

Definition 3.40 (Fourier integral operator(FIO))**.** *We call operators like* (3.9) *Fourier integral operators.*

Note that Example 3.39 is an example of Fourier integral operator with $X=\mathbb{R}_t\times\mathbb{R}^n_x$ and $Y=\mathbb{R}^n_y$.

Definition 3.41 (Pseudodifferential operator)**.** *For the special case of $\varphi(x,y,\theta) = (x-y)\cdot\theta$, $\theta\in\mathbb{R}^n$. We call such Fourier integral operators by pseudodifferential operators, that is,*

$$Au(x) = \iint a(x,y,\theta)e^{i(x-y)\cdot\theta}u(y)\, dy\, d\theta.$$

The name "pseudodifferential" is motivated by the following. Suppose $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha$ is a differential operator, where $a_\alpha\in C^\infty(X)$. A direct computation gives

$$(P(x,D)u)(x) = \frac{1}{(2\pi)^n}\iint \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha e^{i(x-y)\cdot\xi}u(y)\, dy\, d\xi,$$

where

$$a(x,\xi) = \frac{1}{(2\pi)^n}\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in S^m(X\times\mathbb{R}^n).$$

So the pseudodifferential operators are generalizations of differential operators.

Remark 3.42. We will prove later that for a pseudodifferential operator $Au(x)$, we can find a symbol b, independent of y, such that,

$$Au(x) = \iint a(x, y, \theta) e^{i(x-y)\cdot\theta} u(y)\, dy\, d\theta = \iint b(x, \theta) e^{i(x-y)\cdot\theta} u(y)\, dy\, d\theta.$$

Finally, we introduce a theorem.

Theorem 3.43. *Suppose A is an Fourier integral operator.*

If for all $x \in X$, $(y, \theta) \mapsto \varphi(x, y, \theta)$ is a non-degenerate phase function, that is, $d_{y,\theta}\varphi \neq 0$, then

$$A : C_c^\infty(Y) \to C^\infty(X).$$

If for all $y \in X$, $(x, \theta) \mapsto \varphi(x, y, \theta)$ is a non-degenerate phase function, that is, $d_{x,\theta}\varphi \neq 0$, then

$$A : \mathcal{E}'(Y) \to \mathcal{D}'(X).$$

Note that both conditions are satisfied for pseudodifferential operators.

Proof. **For the first part:** Let $\Phi(x, y, \theta) = |d_y\varphi|^2 + |\theta|^2 |d_\theta\varphi|^2$, which is nonzero on $X \times Y \times \dot{\mathbb{R}}^N$ and homogeneous of degree 2. Set $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $\chi \equiv 1$ near 0. Let

$${}^tL = \frac{1 - \chi(\theta)}{i\Phi} \left(\langle \partial_y \varphi, \partial_y \rangle + |\theta|^2 \langle \partial_\theta \varphi, \partial_\theta \rangle \right) + \chi(\theta),$$

then ${}^tL(e^{i\varphi}) = e^{i\varphi}$. Note that L satisfies

$$L = \langle A, \partial_y \rangle + \langle B, \partial_\theta \rangle + c,$$

where $A \in S^{-1}, B \in S^0, c \in S^{-1}$. Thanks to our theory, namely (3.5), we can integrate by parts and get

$$\langle Au, v \rangle = \iiint e^{i\varphi} v(x) L^k(au)\, dx\, dy\, d\theta.$$

Since $L^k(au) \in S_{\rho,\delta}^{m - k\min(\rho, 1-\delta)}$, we know the integral on the right hand side is well-defined. Hence, $Au(x)$ coincides with the smooth function

$$Au(x) = \iint e^{i\varphi} L^k(au)\, dy\, d\theta \in C^\infty(X)$$

since we can choose k any large.

For the second part: We prove this by duality argument. For $A : C_c^\infty(Y) \to \mathcal{D}'(X)$, we define ${}^tA : C_c^\infty(X) \to \mathcal{D}'(Y)$ by

$$\langle {}^tAv, u \rangle = \langle Au, v \rangle, \quad v \in C_c^\infty(X), u \in C_c^\infty(Y).$$

The map ${}^tA$ is continuous by the definition of A and the estimate

$$\left| \langle {}^tAv, u \rangle \right| = |\langle K, v \otimes u \rangle| \leq C \sum_{|\alpha|, |\beta| \leq N} \sup \left| \partial_x^\alpha \partial_y^\beta (uv) \right|.$$

If A is defined by using $K_A \in \mathcal{D}'(X \times Y)$, then ${}^tA$ is defined using $K_{{}^tA}$, where $K_{{}^tA}(y, x) = K_A(x, y)$. Since $K_A = I(a, \varphi)$, then $K_{{}^tA} = I(\widetilde{a}, \widetilde{\varphi})$, where $\widetilde{a}(y, x, \theta) = a(x, y, \theta)$ and $\widetilde{\varphi}(y, x, \theta) = \varphi(x, y, \theta)$.

Now the position of x and y switch, so we can apply the first part of this theorem that we just proved to get

$$K_{{}^tA} : C_c^\infty(X) \to C^\infty(Y).$$

Note that the dual space of smooth functions on an open set is the compactly supported distributions, that is, $(C^\infty(Y))^* = \mathcal{E}'(Y)$. Take $u \in \mathcal{E}'(Y)$, $v \in C_c^\infty(X)$. We define

$$\langle Au, v\rangle := \langle u, {}^tAv\rangle.$$

The extension is unique thanks to the density of inclusion $C_c^\infty(Y) \subset \mathcal{E}'(Y)$. (The density can be shown by convolving it with a C_c^∞ function converging to δ_0 in distributions thanks to convolution mapping property $C_c^\infty * \mathcal{E}' \to C_c^\infty$. See [3, Theorem 5.2.2, 5.2.3].) ☐

3.4. **Stationary phase method and Steepest descent method.** Now we want to "evaluate" the oscillatory integrals.

We study

$$I(\lambda) := \int e^{i\lambda\varphi(x)}a(x)\,dx, \quad a \in C_c^\infty(\mathbb{R}), \varphi \in C^\infty(\mathbb{R};\mathbb{R}), \tag{3.10}$$

$$J(\lambda) := \int e^{-\lambda\psi(x)}a(x)\,dx, \quad a \in C_c^\infty(\mathbb{R}), \psi \in C^\infty(\mathbb{R};\mathbb{R}), \tag{3.11}$$

which corresponds to the stationary phase method and the steepest descent method, respectively.

Theorem 3.44 (Steepest descent)**.** *Suppose $a \in C_c^\infty$ and ψ has a unique non-degenerate minimum at $x_0 \in \operatorname{supp} a$, that is, $\psi'(x_0) = 0, \psi''(x_0) > 0$. Then*

$$J(\lambda) = \int e^{-\lambda\psi(x)}a(x)\,dx \sim e^{-\lambda\psi(x_0)}\left(\left(\frac{2\pi}{\lambda\psi''(x_0)}\right)^{\frac{1}{2}}a(x_0) + b_1\lambda^{-\frac{1}{2}-1} + b_2\lambda^{-\frac{1}{2}-2} + \cdots\right).$$

This means for all N, there exists C such that

$$\left|\int e^{-\lambda\psi(x)}a(x)\,dx - e^{-\lambda\psi(x_0)}\left(\left(\frac{2\pi}{\lambda\psi''(x_0)}\right)^{\frac{1}{2}}a(x_0) + b_1\lambda^{-\frac{1}{2}-1} + \cdots + b_N\lambda^{-\frac{1}{2}-N}\right)\right| \le Ce^{-\lambda\psi(x_0)}\lambda^{-N-\frac{3}{2}}.$$

Proof. Take $\chi \in C_c^\infty(\mathbb{R})$ such that χ is supported very close to x_0. Then

$$\left|\int e^{-\lambda\psi(x)}(1-\chi(x))a(x)\,dx\right| \le e^{-\lambda(\psi(x_0)+\varepsilon)}\int |(1-\chi(x))a(x)|\,dx,$$

which will decay exponentially when λ is large, so we have $J(\lambda) = \int e^{-\lambda\psi(x)}\chi(x)a(x)\,dx + O(e^{-\lambda(\psi(x_0)+\varepsilon)})$. Hence, we only need to consider the integral in a very small interval near x_0. Without loss of generality, we assume $\psi(x_0) = 0$, then using the Taylor remainder formula, we get

$$\psi(x) = \frac{1}{2}(x-x_0)^2\psi_1(x),$$

where $\psi_1(x_0) = \psi''(x_0) > 0$. So we are allowed to take $y = y(x) = (x-x_0)\sqrt{\psi_1(x)}$ near x_0. Since $y'(x_0) \neq 0$, $x = x(y)$ is a change of variable near $y = 0$ and $\varphi(x(y)) = \frac{1}{2}y^2$. We compute

$$\int e^{-\lambda\psi(x)}\chi(x)a(x)\,dx = \int e^{-\frac{\lambda}{2}y^2}b(y)\,dy.$$

Here

$$b(y) = \chi(x(y))a(x(y))\det|\frac{dx}{dy}|, \quad b(0) = a(x_0)(\psi''(x_0))^{-\frac{1}{2}}.$$

One should notice that we need to make a clever choice for χ to make this perfectly well-defined. Since the inverse function only exists locally, we need to choose χ with support sufficiently close to x_0 in which the inverse exists.

Now we apply the Plancherel formula to get

$$\begin{aligned}
&\int e^{-\frac{\lambda}{2}y^2}b(y)\,dy = \frac{1}{2\pi}\int \sqrt{2\pi}\lambda^{-\frac{1}{2}}e^{-\frac{1}{2\lambda}\xi^2}\widehat{b}(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\lambda}}\int e^{-\frac{1}{2\lambda}\xi^2}\widehat{b}(\xi)\,d\xi,\\
=&\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\lambda}}\left(\int \widehat{b}(\xi)\sum_{k=0}^{N-1}\frac{1}{k!}\left(-\frac{1}{2\lambda}\right)^k \xi^{2k}\,d\xi + \left(-\frac{1}{2\lambda}\right)^N\int (\theta\xi)^{2N}\widehat{b}(\xi)\,d\xi,\right),\\
=&\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\lambda}}\left(\sum_{k=0}^{N-1}\frac{1}{k!}\left(-\frac{1}{2\lambda}\right)^k\int \widehat{(D_y^{2k}b)}(\xi)\,d\xi + \left(-\frac{1}{2\lambda}\right)^N\int (\theta\xi)^{2N}\widehat{b}(\xi)\,d\xi,\right),\\
=&\frac{\sqrt{2\pi}}{\sqrt{\lambda}}\sum_{k=0}^{N-1}\frac{1}{k!}\left(-\frac{1}{2\lambda}\right)^k(D^{2k}b)(0) + C_N\lambda^{-N-\frac{1}{2}}\left\|\widehat{D^{2N}b}\right\|_{L^1},\\
=&\frac{\sqrt{2\pi}}{\sqrt{\lambda}}\left(a(x_0)(\psi''(x_0))^{-\frac{1}{2}} + \sum_{k=1}^{N-1}\frac{1}{k!}\left(-\frac{1}{2\lambda}\right)^k((-\partial^2)^k b)(0)\right) + C_N\lambda^{-N-\frac{1}{2}}\left\|\widehat{D^{2N}b}\right\|_{L^1}.
\end{aligned}$$

More precisely, for the remainder term, we have

$$\begin{aligned}
\left\|\widehat{D^{2N}b}\right\|_{L^1} &\le C\|(1+|\xi|^2)\widehat{D^{2N}b}(\xi)\|_{L^\infty} \le C\left(\|\widehat{D^{2N}b}(\xi)\|_{L^\infty} + \|\widehat{D^{2N+2}b}(\xi)\|_{L^\infty}\right)\\
&\le C\left(\|D^{2N+2}b\|_{L^1} + \|D^{2N}b\|_{L^1}\right) \le C\sup_{|\alpha|\le 2N+2}|\partial^\alpha b|,
\end{aligned} \tag{3.12}$$

where the constant in the last inequality depends on supp$b$ since $b \in C_0^\infty$. ☐

As an example and an application, we introduce a proof of the Stirling's formula.

Theorem 3.45 (Stirling's formula)**.** *We have the following approximation for Gamma functions*

$$\Gamma(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}\left(1 + O(\frac{1}{s})\right).$$

Here $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{Z}_+$.

Proof. We study the behavior of

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\,\frac{dt}{t}$$

as $s \to \infty$. By a change of variable $t = sx$, we have

$$\Gamma(s) = s^s\int_0^\infty e^{-s(x-\log x)}\,\frac{dx}{x}.$$

Here our phase function is $\psi(x) = x - \log x$, which has a minimum at $x = 1$ and $\psi''(x) = \frac{1}{x^2} > 0$. Choose $\chi \in C_c^\infty((0,\infty))$ such that $\chi \equiv 1$ near $x = 1$. Then there exists some $\varepsilon > 0$ such that $x - \log x > 1 + \varepsilon$ for all $x \notin \mathrm{supp} a$. Hence,

$$\int_0^1 (1-\chi(x))e^{-s(x-\log x)}\frac{dx}{x} = \int_0^1 (1-\chi(x))e^{-(s-1)(x-\log x)}e^{-x}\,dx = O(e^{-(s-1)(1+\varepsilon)}) = O(e^{-s(1+\varepsilon)})$$

and

$$\int_1^\infty (1-\chi(x))e^{-s(x-\log x)}\frac{dx}{x} = O(e^{-s(1+\varepsilon)}).$$

Now we have

$$\Gamma(s) = s^s\left(\int_0^\infty \chi(x)e^{-s(x-\log x)}\,\frac{dx}{x} + O(e^{-s(1+a)})\right) = s^s e^{-s}\left(\sqrt{2\pi}s^{-\frac{1}{2}} + a_1 s^{-\frac{3}{2}} + \cdots + O(s^{-N-\frac{1}{2}})\right).$$

□

Remark 3.46. We call it by steepest descent method due to the following reason. Note that $-\psi(x)$ will have a unique maximum and going down away from the maximum point.

This method also holds for complex functions. Take $a(z), \psi(z)$ both holomorphic such that $\psi'(0) = 0$. Suppose γ is a contour in the complex plane, if we want to study the integral

$$\int_\gamma a(z)e^{-\psi(z)}\,dz,$$

we need to deform γ to a contour $\widetilde{\gamma}$ such that the harmonic function $\mathrm{Re}\psi(z) \sim t^2$ on $\widetilde{\gamma}$. For example, for $\psi(z) = z^2$, $\mathrm{Re}\psi(z) = x^2 - y^2$, then the good contour will be the x-axis since $\mathrm{Re}\psi(z)$ behave as x^2, which has a non-degenerate minimum. So actually you want to choose a contour on which $-\mathrm{Re}\psi(z)$ has steepest descent.

Now we turn to the stationary phase method. We first consider the case for dimension 1. Suppose $a \in C_c^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R};\mathbb{R})$. We have the following lemma.

Lemma 3.47 (Non-stationary phase lemma)**.** *Suppose* $|\varphi'(x)| > 0$ *on* $\mathrm{supp} a$. *Then for all* N, *there exists* $C_N = C(\mathrm{supp} a, N, \varphi)$ *such that*

$$|I(\lambda)| \leq C_N \sup_{|\alpha|\leq N} |\partial^\alpha a|\lambda^{-N}.$$

Proof. Let ${}^tL = \frac{1}{i|\varphi'|^2}\varphi'\cdot\partial_x$, then $\lambda^{-1}\,{}^tL(e^{i\lambda\varphi}) = e^{i\lambda\varphi}$. Hence,

$$I(\lambda) = \lambda^{-N}\int e^{i\lambda\varphi}(L^N a)(x)\,dx \leq |\mathrm{supp} a|\lambda^{-N}\sup|L^N a| \leq C_N\lambda^{-N}\sup_{|\alpha|\leq N}|\partial^\alpha a|.$$

□

Remark 3.48. This lemma holds for higher dimension case as well with just the same proof.

Theorem 3.49 (Stationary phase theorem for dimension 1)**.** *Suppose* $a \in C_c^\infty(\mathbb{R})$, $\mathrm{supp} a \subset (\alpha,\beta)$, *and* $\varphi \in C^\infty(\mathbb{R};\mathbb{R})$ *has a unique critical point in* (α,β) *which is non-degenerate, that is,* $\varphi'(x_0) = 0, \varphi''(x_0) \neq 0$. *Then*

$$I(\lambda) = e^{i\lambda\varphi(x_0)}\left(\frac{2\pi}{|\varphi''(x_0)|}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}\mathrm{sgn}\varphi''(x_0)}\lambda^{-\frac{1}{2}}(a_0 + a_1\lambda^{-1} + \cdots + a_{N-1}\lambda^{-N+1}) + \lambda^{-N-\frac{1}{2}}S_N,$$

where $a_0 = a(x_0)$ *and*

$$|S_N| \le C_N \sup_{|\alpha|\le 2N+2} |\partial^\alpha a|.$$

Proof. The proof is the same as that of Theorem 3.44. From Lemma 3.47,

$$I(\lambda) = \int \chi(x)a(x)e^{i\lambda\varphi(x)}\,dx + O(\lambda^{-\infty}).$$

Let $\varphi(x) = \varphi(x_0) + \varepsilon\frac{1}{2}(y(x))^2$, where $y(x_0) = 0$, $y'(x_0) = |\varphi''(x_0)|^{\frac{1}{2}} \neq 0$, $\varepsilon = \mathrm{sgn}\varphi''(x_0)$. Without loss of generality, we assume $\varphi(x_0) = 0$. Then

$$I(\lambda) = e^{i\lambda\varphi(x_0)} \int b(y)e^{i\varepsilon\frac{\lambda y^2}{2}}\,dy + O(\lambda^{-\infty}),$$

where $b(y) = a(x(y))\chi(x(y))|\frac{dx}{dy}|$. Note that $b \in C_c^\infty \subset \mathscr{S}$, which implies

$$I(\lambda) = e^{i\lambda\varphi(x_0)}\langle e^{i\varepsilon\frac{\lambda y^2}{2}}, b(y)\rangle_{\mathscr{S}',\mathscr{S}} + O(\lambda^{-\infty}) = \frac{1}{2\pi}\left\langle \widehat{e^{i\varepsilon\frac{\lambda\cdot^2}{2}}}, \widehat{b}\right\rangle_{\mathscr{S}',\mathscr{S}} + O(\lambda^{-\infty}).$$

Since

$$\int e^{i\varepsilon\frac{\lambda y^2}{2}}e^{-\frac{1}{2}\delta y^2}e^{-iy\cdot\xi}\,dy = \int e^{-\frac{1}{2}(\delta - i\varepsilon\lambda)(y+\frac{i\xi}{\delta-i\varepsilon\lambda})^2 - \frac{\xi^2}{2(\delta-i\varepsilon\lambda)}}\,dy = \frac{\sqrt{2\pi}}{\sqrt{\delta - i\varepsilon - \lambda}}e^{-\frac{\xi^2}{2(\delta-i\varepsilon\lambda)}}$$

converges to $\sqrt{\frac{2\pi}{\lambda}}e^{\frac{i\pi}{4}\varepsilon}e^{-\frac{i\varepsilon\xi^2}{2\lambda}}$ in $\mathscr{S}'$ as $\delta \to 0_+$ thanks to the dominated convergence theorem, we have

$$\begin{aligned}
I(\lambda) =& \frac{1}{\sqrt{2\pi\lambda}}e^{\frac{i\pi}{4}\varepsilon}\int_{\mathbb{R}} e^{-\frac{i\varepsilon\xi^2}{2\lambda}}\widehat{b}(\xi)\,d\xi \\
=& \frac{1}{\sqrt{2\pi\lambda}}e^{\frac{i\pi}{4}\varepsilon}\sum_{k<N}\int_{\mathbb{R}}\frac{1}{k!}\left(\frac{-i\varepsilon}{2\lambda}\right)^k \xi^{2k}\widehat{b}(\xi)\,d\xi + \lambda^{-N-\frac{1}{2}}\int O(\xi^{2N})|\widehat{b}(\xi)|\,d\xi \\
=& \frac{\sqrt{2\pi}}{\sqrt{\lambda}}e^{\frac{i\pi}{4}\varepsilon}\sum_{k<N}\frac{1}{k!}\left(\frac{-i\varepsilon}{2\lambda}\right)^k D^{2k}b(0) + \lambda^{-N-\frac{1}{2}}\|\xi^{2N}\widehat{b}(\xi)\|_{L^1}O(1),
\end{aligned}$$

which completes the proof. The remainder term can be further estimate similarly as in (3.12). ∎

To proceed to the case when dimension is bigger than 1, we need a replacement of the statement that

$$\varphi'(x_0) = 0, \varphi''(x_0) \neq 0 \Longrightarrow \varphi(x) = \varphi(x_0) + \varepsilon\frac{1}{2}(y(x))^2, y(x_0) = 0, y'(x_0) = |\varphi''(x_0)|^{\frac{1}{2}}.$$

That is actually a thing that is interesting and important in its own right and goes by a name - Morse Lemma.

Theorem 3.50 (Morse Lemma)**.** *Suppose $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$, $d\varphi(x_0)=0$ and the real symmetric matrix*

$$\varphi''(x_0)=\begin{pmatrix}\partial^2_{x_1x_1}\varphi & \partial^2_{x_1x_2}\varphi & \cdots & \partial^2_{x_1x_n}\varphi\\ \partial^2_{x_2x_1}\varphi & \cdots & \cdots & \partial^2_{x_2x_n}\varphi\\ \vdots & \vdots & \vdots & \vdots\\ \partial^2_{x_nx_1}\varphi & \cdots & \cdots & \partial^2_{x_nx_n}\varphi\end{pmatrix}$$

is non-degenerate, that is, $\det\varphi''(x_0)\neq 0$. Then there exists a C^∞ diffeomorphism ("a change of variable") $\chi : nbhd(x_0)\to nbhd(0)$ such that

$$\varphi\circ\chi^{-1}(y)=\varphi(x_0)+\frac{1}{2}\left(y_1^2+\cdots+y_r^2-y_{r+1}^2-\cdots-y_n^2\right)$$

where $(r,n-r)$ is the signature of $\varphi''(x_0)$, that is, $r=\sharp\{$ positive eigenvalues of $\varphi''(x_0)\}$.

Proof. **Step 1:** Without loss of generality, we assume $x_0=0$, $\varphi(x_0)=0$. Since $\varphi''(0)$ is symmetric, we can diagonalize it as

$$\varphi''(0)={}^tU\Lambda U,\quad {}^tUU=I,\Lambda=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix},\lambda_1,\cdots,\lambda_r>0,\lambda_{r+1},\cdots,\lambda_n<0.$$

We denote $|\Lambda|^{\frac12}=\begin{pmatrix}|\lambda_1|^{\frac12} & & \\ & \ddots & \\ & & |\lambda_n|^{\frac12}\end{pmatrix}$. Let $\widetilde{x}=|\Lambda|^{\frac12}Ux$, then the Taylor formula tells us

$$\begin{aligned}\varphi(x)=&\frac{1}{2}\langle\varphi''(0)x,x\rangle+O(|x|^3)=\frac12\langle\Lambda Ux,Ux\rangle+O(|Ux|^3)\\ =&\frac12(\widetilde{x_1}^2+\cdots+\widetilde{x_r}^2-\widetilde{x_{r+1}}^2-\cdots-\widetilde{x_n}^2)+O(|\widetilde{x}|^3).\end{aligned}$$

Step 2: For simplicity, we replace $\widetilde{x}$ by x. The problem has been reduce to $\varphi(x)=\frac12(x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_n^2)+O(|x|^3)$.

On the other hand,

$$\varphi(x)=\int_0^1(1-t)\partial_t^2\left(\varphi(tx)\right)\,dt=\frac12\sum_{j,k}q_{jk}(x)x_jx_k,$$

where $q_{jk}(x)=2\int_0^1(1-t)\frac{\partial^2\varphi}{\partial x_j\partial x_k}(tx)\,dt$, $q_{jk}=q_{kj}$, $q_{jk}(0)=\frac{\partial^2\varphi}{\partial x_j\partial x_k}(0)$. Set $Q(x)=(q_{jk}(x))_{1\le j,k\le n}$, then

$$\varphi(x)=\frac12\langle Q(x)x,x\rangle, Q(0)=\varphi''(0)=\begin{pmatrix}1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1\end{pmatrix}.$$

Now we want to find A such that $A(0)=I$, $x\mapsto A(x)$ is C^∞ and $\langle x,Q(x)x\rangle=\langle A(x)x,Q(0)A(x)x\rangle$. So we need to solve for A such that $Q(x)={}^tA(x)Q(0)A(x)$. We phrase it so that the implicit function theorem could be applied. Let $\mathrm{Mat}(n;\mathbb{R})$ be the space of all real $n \times n$ matrices, $\mathrm{Symm}(n;\mathbb{R})$ be the space of real symmetric $n \times n$ matrices and consider the map

$$F : \mathrm{Mat}(n;\mathbb{R}) \to \mathrm{Symm}(n;\mathbb{R}), \quad A \mapsto {}^tAQ(0)A.$$

Since $F(A+H) = F(A) + dF_A(H) + o(\|H\|)$, we know

$$dF_A : \mathrm{Mat}(n;\mathbb{R}) \to \mathrm{Symm}(n;\mathbb{R}), \quad H \mapsto {}^tAQ(0)H + {}^tHQ(0)A,$$

then $dF_I(H) = Q(0)H + {}^tHQ(0)$, which is surjective since for all $S \in \mathrm{Symm}(n;\mathbb{R})$, $H = \frac{1}{2}Q(0)^{-1}S$ satisfies that $dF_I(H) = S$. Now we can apply the implicit function theorem (Lemma 3.51) with $M = n^2, N = \frac{n(n+1)}{2}$, then there exists A such that $\langle x, Q(x)x\rangle = \langle A(x)x, Q(0)A(x)x\rangle$. Hence, let $\kappa(x) = A(x)x$, then $\varphi(x) = \frac{1}{2}\langle x, Q(x)x\rangle = \frac{1}{2}\langle \kappa(x), Q(0)\kappa(x)\rangle$, which completes the proof. □

In the proof above, we use the lemma below.

Lemma 3.51 (Implicit function theorem)**.** *Let $F : \mathbb{R}^M \to \mathbb{R}^N$ be a C^1 function such that $F(X_0) = Y_0$. Note that the differential is defined as $dF : \mathbb{R}^M \to L(\mathbb{R}^M, \mathbb{R}^N)$ where $L(\mathbb{R}^M, \mathbb{R}^N)$ is all the linear transformations. If the differential $dF(X_0) : \mathbb{R}^M \to \mathbb{R}^N$ is surjective, then we can solve $F(X(Y)) = Y$ for X such that $X(Y_0) = X_0$.*

Remark 3.52. This is the finite dimentional case. Actually, the implicit function theorem holds for Banach spaces.

And we would use the following Fourier transform in the proof of the theorem below.

Lemma 3.53. *Suppose Q is a symmetric matrix, $\det Q \neq 0$, then*

$$\mathcal{F}\left(e^{\frac{i\lambda\langle Qx,x\rangle}{2}}\right) = \frac{(2\pi)^{\frac{n}{2}}}{|\det Q|^{\frac{1}{2}}} e^{\frac{i\pi}{4}\operatorname{sgn}(Q)} \lambda^{-\frac{n}{2}} e^{\frac{-i\langle Q^{-1}\xi,\xi\rangle}{2\lambda}}.$$

Proof. We diagonalize the matrix Q, then we manipulate it as for the 1 dimensional case n times, which leads to this formula. □

Theorem 3.54 (Stationary phase theorem for higher dimensions)**.** *Suppose $a \in C_c^\infty(X)$, and $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ has a unique critical point in X which is non-degenerate, that is, $\varphi'(x_0) = 0, \varphi''(x_0) \neq 0$. Then*

$$I(\lambda) = e^{i\lambda\varphi(x_0)} \frac{(2\pi)^{\frac{n}{2}}}{|\det \varphi''(x_0)|^{\frac{1}{2}}} e^{\frac{i\pi}{4}\operatorname{sgn}(\varphi''(x_0))} \lambda^{-\frac{n}{2}} (a_0 + a_1\lambda^{-1} + \cdots + a_{N-1}\lambda^{-N+1}) + \lambda^{-N-\frac{n}{2}} S_N,$$

where $a_0 = a(x_0)$ and

$$|S_N| \le C_N \sup_{|\alpha| \le 2N+n+1} |\partial^\alpha a|.$$

More precisely,

$$a_k = \left(A_{2k}(x, D_x)u\right)(x_0),$$

where A_{2k} is a differential operator of order less than or equal to $2k$, $A_0 = I$.

Proof. The proof is the same as the 1 dimensional case. Without loss of generality, we assume $\varphi(x_0) = 0$. From Lemma 3.47 and the change of variables in Morse lemma, we select Q with

$1, -1$ on the diagonal such that $\mathrm{sgn}Q = \mathrm{sgn}\varphi''(x_0)$. Then

$$I(\lambda) = \int \chi(x)a(x)e^{i\lambda\varphi(x)}\,dx + O(\lambda^{-\infty}) = \int e^{i\lambda\frac{1}{2}\langle Qx,x\rangle} b(x)\,dx + O(\lambda^{-\infty}),$$

where $b(x) = a(\kappa^{-1}(x))\chi(\kappa^{-1}(x))|\frac{d\kappa^{-1}}{dx}|$ with $|\frac{d\kappa^{-1}}{dx}(0)| = \frac{1}{|\varphi''(0)|^{\frac{1}{2}}}$. The last equality can be seen from computing the second derivative of $\varphi \circ \kappa^{-1}(y) = \varphi(x_0) + \frac{1}{2}\langle Qy, y\rangle$ in the change of variable in the Morse lemma. Now, Lemma 3.53 gives

$$\begin{aligned}
I(\lambda) &= \frac{1}{(2\pi)^n}\frac{(2\pi)^{\frac{n}{2}}}{|\det Q|^{\frac{1}{2}}} e^{\frac{i\pi}{4}\mathrm{sgn}(Q)}\lambda^{-\frac{n}{2}} \int e^{\frac{-i\langle Q^{-1}\xi,\xi\rangle}{2\lambda}}\widehat{b}(\xi)\,d\xi + O(\lambda^{-\infty})\\
&= \frac{1}{(2\pi)^n}\frac{(2\pi)^{\frac{n}{2}}}{|\det Q|^{\frac{1}{2}}} e^{\frac{i\pi}{4}\mathrm{sgn}(Q)}\lambda^{-\frac{n}{2}} \sum_{k<N}\int \frac{1}{k!}\left(\frac{\langle Q^{-1}\xi,\xi\rangle}{2\lambda i}\right)^k \widehat{b}(\xi)\,d\xi + \lambda^{-\frac{n}{2}-N}\int O(\langle\xi\rangle^{2N})|\widehat{b}(\xi)|\,d\xi + O(\lambda^{-\infty})\\
&= \frac{(2\pi)^{\frac{n}{2}}}{|\det Q|^{\frac{1}{2}}} e^{\frac{i\pi}{4}\mathrm{sgn}(Q)}\lambda^{-\frac{n}{2}} \sum_{k<N}\frac{1}{k!}\left(\left(\frac{\langle Q^{-1}D_x, D_x\rangle}{2\lambda i}\right)^k b\right)(0) + \lambda^{-\frac{n}{2}-N}S_N + O(\lambda^{-\infty}),
\end{aligned}$$

where

$$|S_N(\lambda)| \le C\|\langle\xi\rangle^{2N}\widehat{b}\|_{L^1} \le C\sum_{|\alpha|\le 2N+n+1}\|\partial^\alpha b\|_{L^1} \le C(\mathrm{supp} b)\sum_{|\alpha|\le 2N+n+1}\|\partial^\alpha b\|_{L^\infty},$$

which is a similar estimate as in (3.12).

For the first term in the expansion, it is easy to derive the explicit formula

$$b(0) = \frac{a(x_0)}{|\varphi''(0)|^{\frac{1}{2}}}.$$

□

Let's consider an example with the quadratic form in $\mathbb{R}^{2n}$. This example will be useful for the study of pseudodifferential operators.

Example 3.55. *Let* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $Q = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$, *then* $Q^{-1} = Q$, $\mathrm{sgn}(Q) = 0$. *Since* $\frac{1}{2i}\langle Q^{-1}D_{x,y}, D_{x,y}\rangle = \frac{1}{i}\sum_{j=1}^n \partial_{x_j}\partial_{y_j}$, *Theorem 3.54 gives*

$$\left(\frac{\lambda}{2\pi}\right)^n \int e^{-i\lambda x\cdot y}u(x,y)\,dx\,dy = \sum_{k=0}^{N-1}\frac{1}{k!\lambda^k}\left(\sum_{j=1}^n \frac{1}{i}\partial_{x_j}\partial_{y_j}\right)^k u(0,0) + S_N\lambda^{-N},$$

where $u \in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, $|S_N| \le C\sum_{|\alpha+\beta|\le 2n+1}\sup_{x,y}\left|\partial_x^\alpha\partial_y^\beta(\partial_x\cdot\partial_y)^N u\right|$.

Here is another application of the Fourier transform in Lemma 3.53.

Example 3.56. *We can find a fundamental solution to the Schrodinger equation*

$$\left(i\partial_t + \frac{1}{2}\langle Q^{-1}D_x, D_x\rangle\right)E = \delta_0(t)\delta_0(x).$$

If $Q^{-1} = -2I$, *then it becomes the free Schrodinger equation*

$$(i\partial_t + \Delta)\,E = \delta_0(t)\delta_0(x).$$

Now, suppose the Fourier transform of $E(t,x)$ in x (exists and) is equal to $\widehat{E}(t,\xi)$, then

$$\left(i\partial_t + \frac{1}{2}\langle Q^{-1}\xi,\xi\rangle\right)\widehat{E} = \delta_0(t),$$

which is equivalent to

$$i\partial_t\left(e^{-\frac{i}{2}t\langle Q^{-1}\xi,\xi\rangle}\widehat{E}\right) = \delta_0(t).$$

So we only need to know the fundamental solution to $i\partial_t$, which is the Heaviside function, say

$$e^{-\frac{i}{2}t\langle Q^{-1}\xi,\xi\rangle}\widehat{E} = \frac{1}{i}H(t).$$

Hence,

$$\widehat{E}(t,\xi) = \frac{1}{i}e^{\frac{i}{2}t\langle Q^{-1}\xi,\xi\rangle}H(t)$$

and finally, we apply Lemma 3.53 and get

$$E(t,x) = \frac{1}{i}\frac{|\det Q|^{\frac{1}{2}}}{(2t\pi)^{\frac{n}{2}}}e^{\frac{i\pi}{4}\operatorname{sgn}(Q)}e^{\frac{\langle Qx,x\rangle}{2it}}H(t).$$

3.5. **Pseudodifferential Operators.** Recall that we have defined the pseudodifferential operators in Definition 3.41. Now, we introduce a new notation and only consider the case that $X = Y$.

Definition 3.57. *Let $\rho > 0, \delta < 1$. We say A is a pseudodifferential operator of order $\le m$ if there exists $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ such that for $u \in C_c^\infty(X)$,*

$$Au(x) = \frac{1}{(2\pi)^n}\iint a(x,y,\xi)e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi$$

which is meant as an oscillatory integral. Here $X \subset \mathbb{R}^n$ is an open set. We denote the space of pseudodifferential operators of order $\le m$ by $A \in \Psi^m_{\rho,\delta}(X)$.

Example 3.58. *Here are some examples.*

(1) Let $A := \sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha$ be a differential operator with smooth coefficients, then for this case, $a(x,y,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, which follows from the Fourier inversion formula.

(2) For $(I-\Delta) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, we have an inverse $(I-\Delta)^{-1} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, which gives by the Fourier inversion formula. Note that $(I-\Delta)^{-1} \in \Psi^{-2}_{1,0}$ since $a(x,y,\xi) = (1+|\xi|^2)^{-1}$.

(3) We consider the solution $v = Eu$ to $Pv = u \in \mathcal{E}'(\mathbb{R}^{n+1})$, that is $P(Eu) = u$, where $P = \partial_t - \Delta + 1$. Note that $E \in \Psi^{-1}_{\frac{1}{2},0}(\mathbb{R}^{n+1})$ since $a(t,x,s,y,\tau,\xi) = (i\tau + |\xi|^2 + 1)^{-1} \in S^{-1}_{\frac{1}{2},0}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$, which has been proved in Homework. This is an example in which $\rho \ne 1$.

(4) Suppose $(i\partial_t + \Delta)(Eu) = u$, where $u \in \mathcal{E}'(\mathbb{R}^{n+1})$. Thanks to Example 3.56, we know

$$Eu(x) = \frac{1}{i(2\pi)^n}\int e^{it|\xi|^2}H(t)\widehat{u(t,\xi)}e^{i\xi\cdot x}\,d\xi.$$

In this case, we could write formally that $a_t(x,y,\xi) = e^{it|\xi|^2}H(t)$, where H is the Heaviside function. However, for all t fixed, $\partial_\xi e^{it|\xi|^2} = 2it\xi e^{it|\xi|^2}$, which in particular means *that there is an extra ξ coming out. This is terrible so it is not a symbol and then $P \notin \Psi^*_{*,*}(\mathbb{R}^n)$. Even though P is a nice operator, it is not in any pseudodifferential operator class.*

Now we recall some properties for pseudodifferential operators. Suppose $A \in \Psi^m_{\rho,\delta}(X)$, then Theorem 3.43 tells us

$$A : C_c^\infty(X) \to C^\infty(X),$$
$$A : \mathcal{E}'(X) \to \mathcal{D}'(X).$$

And Theorem 3.35 implies

$$\text{singsupp} K_A \subset \Delta(X \times X) = \{(x,x) : x \in X\}, \tag{3.13}$$

where K_A is the kernel of A, that is,

$$Au(y) = \int K_A(x,y)u(y)\,dy, \quad K_A \in \mathcal{D}'(X \times X),$$

which is meant as the distributional pairing $\langle Au, v\rangle = \langle K_A, v \otimes u\rangle$ for $v \in C_c^\infty(X)$.

Now we introduce the local property first.

Definition 3.59 (Local Property). *We say the continuous map $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$ satisfies the local property if for all $u \in \mathcal{E}'(X)$, $\text{supp}Au \subset \text{supp}u$.*

Note that if A is a differential operator, then A satisfies the local property. In fact, it is necessary.

Theorem 3.60 (Peetre's theorem). *Suppose the linear operator $A : C_c^\infty(X) \to C^\infty(X)$ satisfies the local property, then locally A is a differential operator.*

We follow the proof in [8] in which we need to introduce some lemmas as follows. (One can also find this theorem in [3, Exercise 6.3].) We denote

$$\|u\|_{U,N} = \sum_{|\alpha|\le N} \frac{1}{\alpha!} \sup_{x\in U} |\partial^\alpha u(x)|.$$

Lemma 3.61 (Lemma 1). *Let $f \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha f(0) = 0$ for all $|\alpha| \le N$. Then for all $\varepsilon > 0$, there exists $g \in C^\infty(\mathbb{R}^n)$ which vanishes in a neighborhood of 0 such that $\|f - g\|_{\mathbb{R}^n,N} \le \varepsilon$.*

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \ge 0$ and $\eta \equiv 0$ in $\{|x| \le \frac{1}{2}\}$, $\eta \equiv 1$ if $|x| \ge 1$. For $\delta > 0$, define $g_\delta(x) = \eta(\frac{x}{\delta})f(x)$. Obviously, g_δ vanishes near 0, so it suffices to prove

$$\sup_{x\in\mathbb{R}^n} |\partial^\alpha(g_\delta - f)(x)| \to 0, \text{ as } \delta \to 0$$

for all $|\alpha| \le N$. Note that

$$\sup_{x\in\mathbb{R}^n} |\partial^\alpha(g_\delta - f)(x)| = \sup_{|x|\le\delta} |\partial^\alpha(g_\delta - f)(x)| \le \sup_{|x|\le\delta} |\partial^\alpha f(x)| + \sup_{|x|\le\delta} |\partial^\alpha g_\delta(x)|,$$

where the first term tends to 0. For the second term, we estimate for all $|x| \le \delta$:

$$|\partial^\alpha g_\delta(x)| \le C \sum_{\beta+\gamma=\alpha} \delta^{-|\gamma|} |(\partial^\beta f)(x)| = C \sum_{\beta+\gamma=\alpha} \delta^{-|\gamma|} o(\delta^{N-|\beta|}) = o(\delta^{N-|\alpha|}) = o(1)$$

for $|\alpha| \le N$. This completes the proof. ∎

Lemma 3.62 (Lemma 2). *For all $x_0 \in \mathbb{R}^n$, there exists a small neighborhood of x_0, namely U, an integer $N > 0$ such that*

$$\|Au\|_{\mathbb{R}^n,0} \le C\|u\|_{\mathbb{R}^n,N}$$

for all $u \in C_c^\infty(U - \{x_0\})$.

Proof. Suppose not, then there exists an open set $U_1 \ni x_0$, $\overline{U_1} \Subset U - \{x_0\}$ and $u_1 \in C_c^\infty(U_1)$ such that $\|Au_1\|_{\mathbb{R}^n,0} \ge 4^1\|u_1\|_{\mathbb{R}^n,1}$. Now $U - \overline{U_1}$ is a neighborhood of x_0, then there exists an open set $U_2 \ni x_0$ such that $U_2 \Subset U - \overline{U_1} - \{x_0\}$ and $u_2 \in C_c^\infty(U_2)$ such that $\|Au_2\|_{\mathbb{R}^n,0} \ge 4^2\|u_2\|_{\mathbb{R}^n,2}$. Inductively, we construct a sequence of open sets $\{U_k\}$ such that $\overline{U_k} \cap \overline{U_l} = \varnothing$ for $k \ne l$ and $u_k \in C_c^\infty(U_k)$, $\|Au_k\|_{\mathbb{R}^n,0} \ge 4^k\|u_k\|_{\mathbb{R}^n,k}$.

Let $u = \sum_{k=1}^\infty 2^{-k}\frac{u_k}{\|u_k\|_{\mathbb{R}^n,k}}$, where the sum is convergent in $C^\infty(\mathbb{R}^n)$. So we know $u \in C_c^\infty(U')$ and then $Au \in C_c^\infty(U')$, where U' is a relatively compact neighborhood of U. Furthermore, for all $k \in \mathbb{Z}_+$, $u|_{U_k} = 2^{-k}\|u_k\|^{-1}_{\mathbb{R}^n,k}u_k|_{U_k}$ and $Au|_{U_k} = 2^{-k}\|u_k\|^{-1}_{\mathbb{R}^n,k}Au_k|_{U_k}$. Since $\|Au_k\|_{U_k,0} = \|Au_k\|_{\mathbb{R}^n,0} \ge 4^k\|u_k\|_{\mathbb{R}^n,k}$, we know $\|Au\|_{U_k,0} \ge 2^k$. In particular, $|Au(x_k)| > 2^k$ for some $x_k \in U_k \subset \overline{U'}$, which is compact. This contradicts that $Au \in C_c^\infty(\mathbb{R}^n)$ by extracting a subsequence. ☐

Lemma 3.63 (Lemma 3). *Let U be any open set in $\mathbb{R}^n$, $x_0 \in U$. We assume that*

$$\|Av\|_{\mathbb{R}^n,0} \le C\|v\|_{\mathbb{R}^n,N} \tag{3.14}$$

holds for all $v \in C_c^\infty(U)$. Let $u \in C_c^\infty(U)$ such that $\partial^\alpha u(x_0) = 0$ for all $|\alpha| \le N$, then $Au(x_0) = 0$.

Proof. Without loss of generality, $x_0 = 0 \in U$. From Lemma 1, there exists $u_l \in C_c^\infty(U)$ which vanishes in a neighborhood of 0 such that $\|u_l - u\|_{\mathbb{R}^n,N} \to 0$. By (3.14), $Au_l \to Au$ uniformly on U. Moreover, since $\operatorname{supp}Au_l \subset \operatorname{supp}u_l$, $Au_l(0) = 0$ and hence $Au(0) = 0$. ☐

Now we turn to the proof of Theorem 3.60.

Proof of Theorem 3.60. For all open sets $U \Subset X$, there exists $x_1, \cdots, x_r \in \overline{U}$ such that

$$\|Au\|_{\mathbb{R}^n,0} \le C\|u\|_{\mathbb{R}^n,N}$$

for all $u \in C_c^\infty(U \setminus \cup_j\{x_j\})$, thanks to Lemma 2 and a choice of finite covering of $\overline{U}$.

Now it allows us to apply Lemma 3. For any fixed $u \in C_c^\infty(U \setminus \cup_j\{x_j\})$, all $x_0 \in U \setminus \cup_j\{x_j\}$, let

$$f(x) = u(x) - \sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial^\alpha u(x_0)(x - x_0)^\alpha \in C_c^\infty(U \setminus \cup_j\{x_j\}),$$

then $\partial^\alpha f(x_0) = 0$ for all $|\alpha| \le N$. From Lemma 3, we have

$$0 = Af(x_0) = Au(x_0) - \sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial^\alpha u(x_0)\left(A((x - x_0)^\alpha)\right)(x_0).$$

Let $a_\alpha(x_0) = \frac{1}{\alpha!}\left(A((\cdot - x_0)^\alpha)\right)|_{x_0}$, where we note that $\left(A((\cdot - x_0)^\alpha)\right)|_{x_0}$ is well-defined thanks to the non-increase property of support. Then

$$Au(x) = \sum_{|\alpha|\le N} a_\alpha(x)\partial^\alpha u(x)$$

for all $x \in U \setminus \cup_j\{x_j\}$.

Moreover, for all $v \in C_c^\infty(X)$, for all $x \in U \setminus \cup_j\{x_j\}$, choose $\varphi \in C_c^\infty(U \setminus \cup_j\{x_j\})$, such that $\varphi \equiv 1$ near x. Then

$$Av(x) = A(\varphi v)(x) + A((1-\varphi)v)(x) = A(\varphi v)(x) = \sum_{|\alpha|\le N} a_\alpha(x)\partial^\alpha(\varphi v)(x) = \sum_{|\alpha|\le N} a_\alpha(x)\partial^\alpha v(x),$$

which is independent of φ.

Note that $Av \in C^\infty(X)$ and the right hand side is also smooth in X, we know

$$Av(x) = \sum_{|\alpha|\le N} a_\alpha(x)\partial^\alpha v(x)$$

for all $v \in C_c^\infty(X)$ and all $x \in \overline{U}$. Thanks to the arbitrariness of U, we know A is a differential operator which completes the proof. □

Now we discuss the semilocal property for pseudodifferential operators.

Theorem 3.64 (Semilocal Property). *Let $A \in \Psi^m_{\rho,\delta}(X)$. For all $u \in \mathcal{E}'(X)$, we have*

$$\text{singsupp } Au \subset \text{singsupp } u.$$

Proof. Suppose $x_0 \in X \setminus \text{singsupp } u$. Choose $\varphi \in C_c^\infty(X)$ such that $\varphi \equiv 1$ near x_0, $\psi \in C_c^\infty(X)$ such that $\psi \equiv 1$ near singsupp u and $\text{supp}\varphi \cap \text{supp}\psi = \varnothing$. Note that

$$Au = A(1-\psi)u + A\psi u = A\psi u + g, \quad g \in C^\infty,$$

where the last inequality follows from $(1-\psi)u \in C_c^\infty$. Since $\varphi \equiv 1$ near x_0, it suffices to prove $\varphi Au \in C^\infty$, then Au is smooth near x_0. Moreover, it suffices to show $\varphi A\psi u \in C^\infty$. Since the kernel

$$K_{\varphi A\psi}(x,y) = \varphi(x)K_A(x,y)\psi(y)$$

satisfies $\text{supp}K_{\varphi A\psi} \cap \Delta(X \times X) = \varnothing$, we know $K_{\varphi A\psi} \in C_c^\infty(X \times X)$ thanks to (3.13). Hence, by [3, Corollary 4.1.2], we know $\varphi A\psi u \in C^\infty$ for $u \in \mathcal{E}'(X)$, which completes the proof. □

Definition 3.65. *Let A be a continuous and linear operator $A : C_c^\infty(Y) \to \mathcal{D}'(X)$, then*
(1) A extends to a continuous operator $A : \mathcal{E}'(Y) \to C^\infty(X)$ $\iff$
(2) $K_A \in C^\infty(X \times Y)$.
We say A is a smoothing operator if and only if one of these two conditions holds.

Remark 3.66. In the definition, (2) implies (1) thanks to [3, Corollary 4.1.2], (1) implies (2) thanks to [6, Theorem 5.2.6]. In the proof of (1) ⇒ (2) in the reference, we use a fact that all finite linear combinations of dirac masses are dense in $\mathcal{E}'$, which follows from the fact that dirac masses are dense in C_c^∞ thanks to the Riemann sum definition of integrals and the density of $C_c^\infty \subset \mathcal{E}'$.

Definition 3.67. *We say $A \in \Psi^{-\infty}(X)$ if and only if A is smoothing.*

Proposition 3.68. *We have the following fact $A \in \Psi^{-\infty}(X)$ if and only if $a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$ and A is the pseudodifferential operator with symbol a.*

Proof. ⇐: Suppose $a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$, then $K_A(x,y) = \frac{1}{(2\pi)^n}\int a(x,y,\xi)e^{i(x-y)\cdot\xi}\,d\xi \in C^\infty(X \times X)$.

⇒: Now define

$$a(x,y,\xi) = e^{-i(x-y)\cdot\xi}K_A(x,y)\chi(\xi),$$

where $\chi \in \mathscr{S}(\mathbb{R}^n)$ and $\int \chi(\xi) = (2\pi)^n$. When you differentiate with respect to x and y, you get powers of ξ, but they are all eaten by $\chi(\xi)$, so $a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$. □

Now we want to discuss the composition of the pseudodifferential operators. Let $A \in \Psi^m_{\rho,\delta}(X)$, $B \in \Psi^{m'}_{\rho,\delta}(X)$. We know $A : C_c^\infty(X) \to C^\infty(X)$ and $B : C_c^\infty(X) \to C^\infty(X)$, which is the general case. So we need to impose some assumptions to compose these operators.

Definition 3.69. *A map f is called proper if and only if for all compact sets K, the inverse image $f^{-1}(K)$ is compact.*

To discuss composition of pseudodifferential operators, an important class is properly supported operators. First, we discuss some properties without specializing to pseudodifferntial operators.

Definition 3.70 (Properly supported operator)**.** *We say the linear continuous operator $A : C_c^\infty(Y) \to C^\infty(X)$ is properly supported if and only if*

$$\pi_X|_{\mathrm{supp}K_A} : \mathrm{supp}K_A \to X, \quad \pi(x,y) = x \textit{ is proper},$$

$$\pi_Y|_{\mathrm{supp}K_A} : \mathrm{supp}K_A \to Y, \quad \pi(x,y) = y \textit{ is proper}.$$

Definition 3.71. *Let $C \subset X \times Y$ be a relation. For all $Y' \subset Y$, we define the act of C on Y' by*

$$C(Y') := \{x \in X : \exists (x,y) \in C, y \in Y'\}.$$

Analogously, we define

$$C^{-1}(X') := \{y \in Y : \exists (x,y) \in C, x \in X'\}.$$

Definition 3.72. *If C is a closed subset of $X \times Y$, we say C is proper if the two projections*

$$\pi_X|_C : C \to X, \quad \pi_Y|_C : C \to Y$$

are proper.

Definition 3.73. *Equivalently to Definition 3.70, the linear continuous operator $A : C_c^\infty(Y) \to C^\infty(X)$ is properly supported if and only if $\mathrm{supp}K_A \subset X \times Y$ is proper.*

Lemma 3.74. *For $A : C_c^\infty(Y) \to \mathcal{D}'(X)$, we view $C = \mathrm{supp}K_A$ as a relation, then $\mathrm{supp}Au \subset (\mathrm{supp}K_A)(\mathrm{supp}u)$ for $u \in C_c^\infty(Y)$.*

Proof. We claim $(\mathrm{supp}K_A)(\mathrm{supp}u)$ is closed. By definition, for all $x_n \in C(\mathrm{supp}u)$, $x_n \to x \in X$, there exists $y_n \in \mathrm{supp}u$, $(x_n,y_n) \in C$, Without loss of generality, since C is closed and $\mathrm{supp}u$ is compact, we assume $y_n \to y \in \mathrm{supp}u$ and then $(x_n,y_n) \to (x,y) \in C$ implies $x \in C(\mathrm{supp}u)$. Therefore it follows that $C(\mathrm{supp}u)$ is closed.

Now, for all $x_0 \notin (\mathrm{supp}K_A)(\mathrm{supp}u)$, there exists a neighborhood $U \ni x_0$ such that $U \cap (\mathrm{supp}K_A)(\mathrm{supp}u) = \varnothing$. For all $\phi \in C_c^\infty(U)$, we know $\langle Au, \phi\rangle = \langle K_A, \phi \otimes u\rangle = 0$, which implies $x_0 \notin \mathrm{supp}Au$. □

From Lemma 3.74, we know that if A is properly supported, then A is continuous $C_c^\infty(Y) \to \mathcal{E}'(X)$.

Theorem 3.75. *If $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ is properly supported, then it uniquely extends to $A : C^\infty(Y) \to \mathcal{D}'(X)$ and $A : C_c^\infty(Y) \to \mathcal{E}'(X)$.*

Proof. The second part has been proved in the previous discussion.

For the first part, we want to define Au on all open sets $\widetilde{X}$ such that $\widetilde{X} \Subset X$, for all $u \in C^\infty(Y)$.

There exists $\chi \in C_c^\infty(Y)$ such that $\chi = 1$ on

$$\pi_Y \left(\left(\pi_X|_{\mathrm{supp} K_A}\right)^{-1} \left(\overline{\widetilde{X}}\right)\right) = C^{-1}(\overline{\widetilde{X}}),$$

which is a compact set since it is the continuous image of a compact set thanks to the fact that A is properly supported.

Now for all $v \in C_c^\infty(\widetilde{X})$, we define

$$\langle Au, v\rangle := \langle A(\chi u), v\rangle = \langle K_A, v \otimes (\chi u)\rangle,$$

which is independent of χ, which completes the proof of existence.

The definition is independent of the choice of χ. Suppose $\widetilde{\chi}$ has the same property, then $((\mathrm{supp} K_A)\, \mathrm{supp}(\chi - \widetilde{\chi})) \cap \widetilde{X} = \varnothing$, where $\mathrm{supp} K_A$ acts as a relation, which implies $\langle K_A, v \otimes ((\chi - \widetilde{\chi})u)\rangle = 0$ for $u \in C^\infty(Y)$, $v \in C_v^\infty(\widetilde{X})$.

Moreover, the uniqueness follows from the density of C^∞ in C^∞ in the topology of C^∞. ☐

Let's now go to specialize to pseudodifferential operators. Suppose $A \in \Psi^m_{\rho,\delta}(X)$ is properly supported, then this implies

$$\begin{aligned} A : C_c^\infty(X) \to C_c^\infty(X), \quad C^\infty(X) \to C^\infty(X), \\ A : \mathcal{E}'(X) \to \mathcal{E}'(X), \quad \mathcal{D}'(X) \to \mathcal{D}'(X), \end{aligned} \tag{3.15}$$

thanks to Theorem 3.43 and Theorem 3.75.

Now suppose $B \in \Psi^{m'}_{\rho,\delta}(X)$ not necessarily properly supported, then

$$\begin{aligned} A \circ B :& C_c^\infty(X) \xrightarrow{B} C^\infty(X) \xrightarrow{A} C^\infty(X), \quad \mathcal{E}'(X) \xrightarrow{B} \mathcal{D}'(X) \xrightarrow{A} \mathcal{D}'(X), \\ B \circ A :& C_c^\infty(X) \xrightarrow{A} C_c^\infty(X) \xrightarrow{B} C^\infty(X), \quad \mathcal{E}'(X) \xrightarrow{A} \mathcal{E}'(X) \xrightarrow{B} \mathcal{D}'(X). \end{aligned}$$

The theorem below tells us in some sense, each pseudodifferential operator is properly supported modulo smoothing operators.

Theorem 3.76. *Suppose $A \in \Psi^m_{\rho,\delta}(X)$, then $A = A_1 + A_2$, where $A_1 \in \Psi^m_{\rho,\delta}(X)$ is properly supported and A_2 is smoothing, that is, $A_2 \in \Psi^{-\infty}(X)$.*

The proof relies on the following lemma.

Lemma 3.77. *There exists $\chi \in C^\infty(X \times X)$ such that $\chi \equiv 1$ near $\Delta(X \times X)$ and $\mathrm{supp}\chi$ is proper.*

Proof. If $X = \mathbb{R}^n$, the proof is quite easy. Set $\chi(x, y) = \widetilde{\chi}(x - y)$, where $\widetilde{\chi} \in C_c^\infty(\mathbb{R}^n)$ and $\widetilde{\chi} \equiv 1$ near 0, then one can check it is as desired. (This is enough at least intuitively.)

Now for the general case, we need to use partitions of unity, see [3, Formula 1.4.5, p.13]. Let $\varphi_j(x) \in C_c^\infty(X)$ be a locally finite partition of unity, that is, $1 = \sum_{j=0}^\infty \varphi_j(x)$ and if $K \subset X$ is a compact set then there are only finitely many of the φ_j with $K \cap \mathrm{supp}\varphi_j \neq \varnothing$. Then, $1 = \sum_j \sum_k \varphi_j(x)\varphi_k(y)$ is a locally finite partition of unity on $X \times X$. We put

$$\chi(x, y) = \sum_{\mathrm{supp}\varphi_j \cap \mathrm{supp}\varphi_k \neq \varnothing} \varphi_j(x)\varphi_k(y),$$

which is equal to 1 near the diagonal.

Moreover, if $C = \mathrm{supp}\chi$ is viewed as a relation and if $K \subset X$ is compact, then

$$C(K) = \pi_Y\left((\pi_X|_{\mathrm{supp}\chi})^{-1}(K)\right) \subset \cup \mathrm{supp}\varphi_j,$$

where the union is taken over all j such that there exists $k = k(j)$ with $\mathrm{supp}\varphi_j \cap \mathrm{supp}\varphi_k \neq \varnothing$ and $\mathrm{supp}\varphi_k \cap K \neq \varnothing$. In particular, the union is a finite union, so $\cup\mathrm{supp}\varphi_j$ is compact, which implies that the closed set $C(K)$ is compact. Similarly, $C^{-1}(K)$ is compact, hence C is proper. ☐

Proof of Theorem 3.76. Let

$$A_1 u = \iint e^{i(x-y)\cdot\xi} a(x, y, \xi)\chi(x, y)u(y)\, dy\, d\xi,$$

$$A_2 u = \int K_A(x, y)(1 - \chi(x, y))u(y)\, dy.$$

Since singsupp $K_A \subset \Delta(X \times X)$, the kernel for A_2 is $K_A(x, y)(1 - \chi(x, y)) \in C^\infty$, which implies $A_2 \in \Psi^{-\infty}(X)$. Moreover, $K_{A_1}(x, y) = K_A(x, y)\chi(x, y)$, then $\mathrm{supp}K_{A_1} = \mathrm{supp}K_A \cap \mathrm{supp}\chi$. Since $\mathrm{supp}K_A$ is closed, $\mathrm{supp}\chi$ is proper, we know $\mathrm{supp}K_{A_1}$ is proper, that is, A_1 is properly supported. ☐

Now we introduce a fundamental theorem, which will be very useful later.

Theorem 3.78. *Suppose $A \in \Psi^m_{\rho,\delta}(X)$ is properly supported and $\rho > \delta$. Then*
(1) $b(x, \xi) := e^{-ix\cdot\xi}A(e^{i\bullet\cdot\xi}) \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$,
(2) $b(x, \xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\left(\partial_\xi^\alpha D_y^\alpha a(x, y, \xi)\right)|_{y=x}$,
(3) For $u \in C_c^\infty(X)$, $Au(x) = \frac{1}{(2\pi)^n}\int b(x, \xi)e^{ix\cdot\xi}\widehat{u}(\xi)\, d\xi$.

Remark 3.79. The asymptotic sum above is defined by regrouping terms with the same value of $|\alpha|$ and we notice that $\partial_\xi^\alpha D_y^\alpha a(x, y, \xi) \in S^{m-|\alpha|(\rho-\delta)}_{\rho,\delta}$, where $m - |\alpha|(\rho - \delta) \to -\infty$ as $|\alpha| \to \infty$ as in the assumption in Theorem 3.21.

Proof. **Step 1:** Firstly, we assume $A \in \Psi^{-\infty}$ and A is properly supported, then it suffices to prove the first conclusion since the second and the third conclusions are trivially valid.

Thanks to Proposition 3.68, we have $a \in S^{-\infty}$. In order to prove $b(x, \xi) \in S^{-\infty}(X \times \mathbb{R}^n)$, we need to consider all $\widetilde{X} \Subset X$, then take $\chi_{\widetilde{X}} \in C_c^\infty$ as in Theorem 3.75, and we know from Theorem 3.75 that $b(x, \xi) := e^{-ix\cdot\xi}A(e^{i\bullet\cdot\xi})$ is defined by

$$b(x, \xi) = \frac{1}{(2\pi)^n}\iint a(x, y, \theta)e^{i(x-y)\cdot(\theta-\xi)}\chi_{\widetilde{X}}(y)\, dy\, d\theta$$

for all $x \in \widetilde{X}$, then we can use

$$\xi^\alpha e^{i(x-y)\cdot(\theta-\xi)} = D_y^\alpha e^{i(x-y)\cdot(\theta-\xi)}, \; D_x^\alpha e^{i(x-y)\cdot(\theta-\xi)} = (-1)^{|\alpha|} D_y^\alpha e^{i(x-y)\cdot(\theta-\xi)},$$
$$D_\xi^\beta e^{i(x-y)\cdot(\theta-\xi)} = (-1)^{|\beta|} D_\xi^\beta e^{i(x-y)\cdot(\theta-\xi)},$$

to integrate by parts and conclude.

Step 2: Hence, in the following steps, we can introduce a cut-off $\chi(x,y)$ as in Lemma 3.77 thanks to Step 1. (In other words, we write $A = A_1 + A_2$ as in Theorem 3.76. In Step 1, we show for A_2, where the properly supported property of $A_2 \in \Psi^{-\infty}$ follows from the assumption that $A = A_1 + A_2$ is properly supported. A_1 is automatically properly supported by Theorem 3.76. So we focus on A_1 in the following steps.) After introducing this cut-off $\chi(x,y)$, one notes that for all $\widetilde{X} \Subset X$, all $x \in \widetilde{X}$, there exists $\widetilde{Y} \Subset Y$ such that $\operatorname{supp} a(x,\cdot,\theta) \subset \widetilde{Y}$. Thus the following integral in y is well-defined in

$$b(x,\xi) = \frac{1}{(2\pi)^n} \int a(x,y,\theta) e^{i(x-y)\cdot(\theta-\xi)} \, dy \, d\theta,$$

and it makes sense as an iterated integral by integration by parts using

$$\frac{1}{1+|\theta-\xi|^2}(1-(\theta-\xi)\cdot D_y) e^{-iy(\theta-\xi)} = e^{-iy(\theta-\xi)}$$

and obtain the following fact $\int a(x,y,\theta) e^{-iy\cdot(\theta-\xi)} \, dy = O(\langle\theta-\xi\rangle^{-\infty})$, which implies $b(x,\xi) \in C^\infty(\widetilde{X} \times \mathbb{R}^n)$. By the arbitrariness of $\widetilde{X}$, we know $b(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$.

Step 3: We consider the phase function $\Phi(y,\theta) = (x-y)\cdot(\xi-\theta)$ although it is not homogeneous. We check the Hessian by computing $\Phi_y = -(\xi-\theta), \Phi_\theta = -(x-y)$, and $\Phi'' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, which is non-degenerate with $\operatorname{sgn}\Phi'' = 0$ and $\det \Phi'' = 1$. These are just the conditions needed for applying stationary phase method.

However, note that θ is being integrated over a non-compact set. To reduce it to the case in which we can apply stationary phase, we introduce the following cut-off functions. We choose $\chi \in C_c^\infty([0,+\infty);[0,1])$ such that $\chi(t) \equiv 1$ when $t \le \frac{1}{3}$ and $\chi(t) \equiv 0$ when $t \ge \frac{1}{2}$. We write

$$b_2(x,\xi) = \int a(x,y,\theta) \left(1 - \chi(\frac{|\theta-\xi|}{|\xi|})\right) e^{i(x-y)\cdot(\theta-\xi)} \, dy \, d\theta, \tag{3.16}$$

Since we already showed $b(x,\xi)$ is smooth, we only need to consider $|\xi| \gg 1$. The integrand in (3.16) does not vanish when $|\theta-\xi| > \frac{1}{3}|\xi| > \frac{1}{3}$, and in particular, it does not vanish when $|\theta-\xi| \sim 1 + |\theta| + |\xi|$. Indeed, $\frac{4}{3}|\theta-\xi| > |\theta-\xi| + |\xi| \ge |\theta|$ and $|\theta-\xi| \le |\theta| + |\xi|$.

Let ${}^tL = \frac{1}{|\theta-\xi|^2}(\xi-\theta)\cdot D_y$, then ${}^tLe^{i\Phi} = -e^{i\Phi}$, and we are allowed to do integration by parts in y since the integral in y is over a compact set when x is over a compact set. Hence, for $x \in \widetilde{X} \Subset X$,

$$b_2(x,\xi) = \int L^N \left(a(x,y,\theta) \left(1 - \chi(\frac{|\theta-\xi|}{|\xi|})\right)\right) e^{i(x-y)\cdot(\theta-\xi)} \, dy \, d\theta,$$

where for all $x \in \widetilde{X} \Subset X$,

$$\left| L^N \left(a(x,y,\theta) \left(1 - \chi(\frac{|\theta-\xi|}{|\xi|})\right)\right)\right| \le C \frac{\langle\theta\rangle^{|m|+N\delta}}{(1+|\theta|+|\xi|)^N} \lesssim \frac{\langle\theta\rangle^{|m|+N\delta}}{\langle\theta\rangle^{|m|+N\delta+n+1}\langle\xi\rangle^{-|m|-n-1+N(1-\delta)}},$$

where in the first step we use $|\theta - \xi| \sim 1 + |\theta| + |\xi|$ and in the second step we use Peetre's lemma([1, Lemma 6.5.6]). This implies $b_2(x, \xi) = O(\langle\xi\rangle^{-M})$ for all $M > 0$ by taking N sufficiently large in the equation above with $M = -|m| - n - 1 + N(1 - \delta)$. Moreover, we have the same type of estimates for derivatives of b_2, so we conclude that $b_2 \in S^{-\infty}(X \times \mathbb{R}^n)$.

Step 4: It remains to study b_1 and we would like to reduce it to a form to which we can apply the method of stationary phase

$$\begin{aligned} b_1(x, \xi) &= \frac{1}{(2\pi)^n} \iint a(x, y, \theta)\chi\left(\frac{|\theta - \xi|}{|\xi|}\right) e^{i(x-y)\cdot(\theta-\xi)}\, dy\, d\theta \\ &= \frac{\lambda^n}{(2\pi)^n} \iint a(x, y, \lambda(\omega + \sigma))\chi(|\sigma|)e^{i\lambda(x-y)\cdot\sigma}\, dy\, d\sigma \\ &= \frac{\lambda^n}{(2\pi)^n} \iint a(x, x + s, \lambda(\omega + \sigma))\chi(|\sigma|)e^{-i\lambda s\cdot\sigma}\, ds\, d\sigma, \end{aligned} \tag{3.17}$$

where we make the change of variable $\theta = \xi + \sigma\lambda$, $\sigma \in \mathbb{R}^n$, $\xi = \lambda\omega$, $\omega \in \mathbb{S}^{n-1}$. By the properly supported assumption of a in (x, y), we know that for $x \in \widetilde{X} \Subset X$, the integral in s is over a compact set, and the integral in (3.17) is integrated over compact sets both in s and σ thanks to the cut-off function.

We can then apply the result in Example 3.55 to (3.17) and get

$$\begin{aligned} b_1(x, \xi) &= \sum_{k=0}^{N-1} \frac{1}{k!\lambda^k} \left[\left(\sum_{j=1}^{n} \partial_{\sigma_j} D_{s_j}\right)^k (a(x, x + s, \lambda(\omega + \sigma)))\right]\Big|_{s=0,\sigma=0} + S_N \\ &= \sum_{|\alpha|<N} \frac{\lambda^{-|\alpha|}}{\alpha!}\partial_\sigma^\alpha D_s^\alpha \left[a(x, x + s, \lambda(\omega + \sigma))\right]\Big|_{s=0,\sigma=0} + S_N \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\theta^\alpha D_y^\alpha a(x, y, \theta)|_{y=x,\theta=\xi} + S_N. \end{aligned}$$

Note that $\partial_\theta^\alpha D_y^\alpha a(x, y, \theta)|_{y=x,\theta=\xi} \in S^{m-(\rho-\delta)|\alpha|}$ with the strictly decreasing order when $\rho > \delta$, so we only need to check the order of the remainder S_N is smaller:

$$\begin{aligned} |S_N(\lambda)| \le& C_N\lambda^{-N} \sum_{|\alpha+\beta|\le 2n+1, |s|\le C_{\widetilde{X}}, |\sigma|\le\frac{1}{2}} \left|\partial_s^\alpha\partial_\sigma^\beta \left(\partial_s \cdot \partial_\sigma\right)^N (a(x, x + s, \lambda(\omega + \sigma)\chi(|\sigma|))\right| \\ \le& C_N \sum_{|\alpha+\beta|\le 2n+1} \lambda^{-N}\lambda^{m+\delta(|\alpha|+N)+(1-\rho)(|\beta|+N)} \le C_N \sum_{|\alpha+\beta|\le 2n+1} \lambda^{-N(\rho-\delta)}\lambda^{m+\delta|\alpha|+(1-\rho)|\beta|} \\ \le& C_N\lambda^{m+2n+1-N(\rho-\delta)} \le C_N\langle\xi\rangle^{m+2n+1-N(\rho-\delta)}, \end{aligned}$$

though the estimate is not quite good but actually it is sufficiently to show the asymptotic expansion for b. We summarize this in Lemma 3.80 below. So now we know $b \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} \left(\partial_\xi^\alpha D_y^\alpha a(x, y, \xi)\right)|_{y=x}$.

Step 5: Thanks to the Fourier inversion formula, $u(x) = \frac{1}{(2\pi)^n} \int \widehat{u}(\xi)e^{ix\cdot\xi}\, d\xi$ for all $u \in C_c^\infty(X)$, and this integral can be approximated by a sequence of Riemann sums

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{2\pi}\right)^n \sum_{\nu\in(\varepsilon\mathbb{Z})^n} e^{ix\cdot\nu}\widehat{u}(\nu),$$

which converges to u in $C^\infty(X)$. Since $A : C^\infty(X) \to C^\infty(X)$ is continuous, we get

$$Au(x) = \lim_{\varepsilon\to 0} A(u_\varepsilon)(x) = \frac{1}{(2\pi)^n}\int A(e^{i\bullet\cdot\xi})\widehat{u}(\xi)\,d\xi = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi} b(x,\xi)\widehat{u}(\xi)\,d\xi,$$

which completes the proof. □

Lemma 3.80. *Suppose $a = \sum_{j<M} a_j\lambda^{-j} + S_M$ and there exists some $\rho > 0, N_0 \in \mathbb{R}$ such that $|S_M| \le C_M\lambda^{N_0-\rho M}$ for all $M > 0$, then we know*

$$\left| a - \sum_{j<N} a_j \lambda^{-j} \right| \le \widetilde{C_N}\lambda^{-N}$$

for all N, that is, $a \sim a_0 + a_1\lambda^{-1} + \cdots + a_k\lambda^{-k} + \cdots$.

Proof. We write

$$a = \sum_{j<N} a_j\lambda^{-j} + \sum_{N\le j<M} a_j\lambda^{-j} + S_M,$$

then we can obtain a new estimate for

$$R_N := \sum_{N\le j<M} a_j\lambda^{-j} + S_M \le C_{N,M}\lambda^{-N} + C_M\lambda^{N_0-\rho M} \le \widetilde{C_N}\lambda^{-N}$$

if we choose M sufficiently large such that $M > \frac{1}{\rho}(N_0+N)$, which completes the proof. □

Corollary 3.81 (Adjoint)**.** *Let $A \in \Psi^m_{\rho,\delta}(X), \rho > \delta$ be properly supported and $Au(x) = \int a(x,\xi)e^{ix\cdot\xi}\widehat{u}(\xi)\,d\xi$. Then the adjoint $A^* : C_c^\infty(X) \to \mathcal{D}'(X)$ defined by $\langle Au, v\rangle = \langle u, A^*v\rangle$, $u, v \in C_c^\infty(X)$ satisfies $A^* \in \Psi^m_{\rho,\delta}(X)$ and*

$$A^*u(x) = \int a^*(x,\xi)e^{ix\cdot\xi}\widehat{u}(\xi)\,d\xi,$$

where $a^*(x,\xi) \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{a(x,\xi)}$.

Proof. Since $K_A(x,y) = \int a(x,\xi)e^{i(x-y)\cdot\xi}\,d\xi$, we have

$$K_{A^*}(x,y) = \overline{K_A(y,x)} = \int \overline{a(y,\xi)}e^{i(x-y)\cdot\xi}\,d\xi, \tag{3.18}$$

which is the previous form that $K_{A^*}(x,y) = \int c(x,y,\xi)e^{i(x-y)\cdot\xi}\,d\xi$, but here $c(x,y,\xi) = \overline{a(y,\xi)}$ is independent of x. Since A is properly supported, we know suppK_A is proper, then A^* is also properly supported thanks to (3.18). Now, we set

$$a^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\left(\partial_\xi^\alpha D_y^\alpha c(x,y,\xi)\right)|_{y=x} \sim \sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)}.$$

Then Theorem 3.78 implies

$$A^*u = \int a^*(x,\xi)e^{ix\cdot\xi}\widehat{u}(\xi)\,d\xi,$$

which completes the proof. □

Definition 3.82. *Suppose $A \in \Psi^m_{\rho,\delta}(X)$, then we write $A = A_1 + A_2$ as in Theorem 3.76, where A_1 is properly supported and $A_2 \in \Psi^{-\infty}$. We call*

$$\sigma_A(x,\xi) := b(x,\xi) = e^{-ix\cdot\xi}A_1(e^{i\bullet\cdot\xi})$$

the full symbol of A, and thanks to Theorem 3.78, we have

$$Au(x) = \frac{1}{(2\pi)^n}\int \sigma_A(x,\xi)e^{ix\cdot\xi}\widehat{u}(\xi)\,d\xi + A_2u(x),$$

and σ_A is unique modulo $S^{-\infty}(X\times\mathbb{R}^n)$.

Then from the corollary above, for all $A \in \Psi^m_{\rho,\delta}(X)$, A^* is also in $\Psi^m_{\rho,\delta}(X)$ and

$$\sigma_{A^*}(x,\xi) \sim \sum \frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_x\overline{\sigma_A(x,\xi)}.$$

Now we have the following theorem for composition.

Theorem 3.83 (Composition)**.** *Suppose $A \in \Psi^m_{\rho,\delta}(X), B \in \Psi^{m'}_{\rho,\delta}(X)$ and either A or B are properly supported, then $A\circ B \in \Psi^{m+m'}_{\rho,\delta}(X)$ and*

$$\begin{aligned}\sigma_{A\circ B}(x,\xi) &\sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi\sigma_A(x,\xi)D^\alpha_x\sigma_B(x,\xi)\\ &= \sigma_A(x,\xi)\sigma_B(x,\xi) + \sum_{|\alpha|=1}\partial^\alpha_\xi\sigma_A(x,\xi)D^\alpha_x\sigma_B(x,\xi) + S^{m+m'-2(\rho-\delta)}_{\rho,\delta}.\end{aligned} \tag{3.19}$$

We denote $\sigma_{A\circ B}$ by $\sigma_{A\circ B} := \sigma_A\sharp\sigma_B$.

Proof. Since at least one of A, B is properly supported, we know the composition is well-defined thanks to (3.15). We only give a proof for the case that B is properly supported and the other case is similar.

We can write

$$Au(x) = \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}a(x,\xi)\chi(x,y)u(y)\,dy\,d\xi + A_2u(x),$$

where A_2 is an element of $\Psi^{-\infty}$ and $A_2\circ B$ is a continuous operator $\mathcal{E}'\to C^\infty$ with a kernel and hence in $\Psi^{-\infty}(X)$. In the equation above, $\chi(x,y)$ is defined in Lemma 3.77 and $a(x,\xi)\sim\sigma_A(x,\xi)$.

We may also assume that

$$Bu(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}b(x,\xi)\widehat{u}(\xi)\,d\xi, \quad u\in C^\infty_c(X),$$

where the integral can be approximated by a sequence of Riemann sums converging in $C^\infty(X)$ like the proof of Theorem 3.78, and $b(x,\xi)\sim\sigma_B(x,\xi)$. Then for $u\in C^\infty_c(X)$, we can obtain that

$$A\circ Bu(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}c(x,\xi)\widehat{u}(\xi)\,d\xi + A_2\circ B,$$

where $c(x,\theta) = e^{-ix\cdot\theta}A(b(\cdot,\theta)e^{i\bullet\cdot\theta})$, $A_2 \circ B \in \Psi^{-\infty}(X)$. As in the proof of Theorem 3.78, we know

$$c \in S^{m+m'}_{\rho,\delta}, \quad c \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi),$$

which completes the proof. (One can just view the symbol here as $a(x,\xi)b(y,\theta)$ and apply the theorem of finding the complete symbol.) □

3.6. **Change of Variables.**

Theorem 3.84 (Change of variables)**.** *Suppose $\kappa : X \to \widetilde{X}$ is a diffeomorphism, that is, κ is C^∞ and $\kappa^{-1} : \widetilde{X} \to X$ is C^∞. For $x \in X$, we denote $\kappa(x) = \widetilde{x} \in \widetilde{X}$.*

Let $\widetilde{P}(\widetilde{x}, D_{\widetilde{x}}) = \sum_{|\alpha|\le m} a_\alpha(\widetilde{x}) D^\alpha_{\widetilde{x}}$ be a differential operator on $\widetilde{X}$, and then we can define a differential operator on X by

$$Pu(x) = \kappa^* \widetilde{P}\left((\kappa^{-1})^* u\right),$$

for $u \in C^\infty(X)$, where $\kappa^* u(x) = u(\kappa(x))$, $\kappa^* : C^\infty(\widetilde{X}) \to C^\infty(X)$. And P is indeed a differential operator given by

$$P(x, D_x) = \sum_{|\alpha|\le m} a_\alpha(\kappa(x)) \left[({}^t\kappa'(x))^{-1} D_x\right]^\alpha.$$

Proof. Note that $D_{\widetilde{x_j}} = \sum_k \frac{\partial x_k}{\partial \widetilde{x_j}} D_{x_k}$ and $(\kappa^{-1})' = (\kappa')^{-1} = \left(\frac{\partial x}{\partial \widetilde{x}}\right)$, so we can rewrite it as $D_{\widetilde{x}} = {}^t(\kappa')^{-1} D_x$, which implies that

$$P(x, D_x) = \sum a_\alpha(\kappa(x)) \left[({}^t\kappa'(x))^{-1} D_x\right]^\alpha.$$

□

In the theorem above, if we only look at the highest order term, we have a nice formula that

$$p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(\kappa(x)) \left[({}^t\kappa'(x))^{-1}\xi\right]^\alpha = \widetilde{p}(\kappa(x), ({}^t\kappa'(x))^{-1}\xi).$$

Example 3.85. *We consider an example in 1 dimension. Let $X, \widetilde{X}$ be two disjoint intervals and κ be a diffeomorphism between intervals. Let $\widetilde{P} = D^2_{\widetilde{x}}$, then $\widetilde{P}(x,\xi) = \widetilde{\xi}^2$, then*

$$P(x, D_x) = \left[{}^t(\kappa')^{-1} D_x\right]^2 = {}^t(\kappa')^{-1} D_x \left({}^t(\kappa')^{-1} D_x\right) = {}^t(\kappa')^{-2} D_x^2 + i\,{}^t(\kappa')^{-2}\,{}^t(\kappa'')^{-1} D_x.$$

And we can drop the transpose since the dimension is 1 now. So $P(x,\xi) = (\kappa')^{-2}\xi^2 + i(\kappa')^{-2}(\kappa'')^{-1}\xi$. Note that $P(x,\xi)$ is more complicated than $\widetilde{P}(x,\xi)$, but if we look for the highest order terms of $P, \widetilde{P}$ and denote them by $p, \widetilde{p}$, we have

$$p(x,\xi) = (\kappa')^{-2}\xi^2 = \widetilde{p}(\kappa(x), (\kappa'(x))^{-1}\xi).$$

Here we present a theorem whose proof can be found in [4]. The proof is a little bit involved but follows the same idea before.

Theorem 3.86. *Let $\kappa : X \to \widetilde{X}$ be a diffeomorphism. Suppose $\widetilde{A} \in \Psi^m_\rho(\widetilde{X}) := \Psi^m_{\rho,1-\rho}(\widetilde{X})$ for $\rho > \frac{1}{2}$, then $A := \kappa^* \widetilde{A} (\kappa^{-1})^* \in \Psi^m_\rho(X)$ and $\sigma_A(x,\xi) = \sigma_{\widetilde{A}}(\kappa(x), ({}^t\kappa')^{-1}\xi)$ mod $S^{m-(2\rho-1)}_{\rho,1-\rho}$.*

Remark 3.87. The choice of $\delta = 1-\rho$ can be seen from the following computation with $n=1$. For $\tilde{a} \in S^m_{\rho,\delta}(\tilde{X} \times \mathbb{R}^N)$, then we want to know $a(x,\xi) = \tilde{a}(\kappa(x), (\kappa'(x))^{-1}\xi)$ in which symbol class it is. When we apply ∂_x to it, it falls on $\tilde{\xi}$ as well :

$$\partial_x a = \kappa'(x)\partial_{\tilde{x}}\tilde{a} - (\kappa'(x))^{-2}\kappa''(x)\xi\partial_{\tilde{\xi}}\tilde{a},$$

where the first term is bounded by $\langle\xi\rangle^{m+\delta}$ while the second term is bounded by $\langle\xi\rangle^{m+(1-\rho)}$.

So if we define the principal symbol of A as $[\sigma_A(x,\xi)] \in S^m_{\rho,\delta}/S^{m-(2\rho-1)}_{\rho,1-\rho}$, which is in an invariantly defined class. And we have some fancy language for this: The symplectic lift of the diffeomorphism κ is defined by $\mathcal{K} : X \times \mathbb{R}^n \to \widetilde{X} \to \mathbb{R}^n$, $\mathcal{K}(x,\xi) = (\kappa(x), {}^t(\kappa'(x))^{-1}\xi)$. This invariance helps us to define symbols on manifolds.

3.7. **Characteristic set and Ellipticity.** Let $P \in \Psi^m_{1,0}(X)$ and denote for simplicity its symbol by $P(x,\xi)$.

Definition 3.88. *We say $P \in \Psi^m_{1,0}(X)$ is non-characteristic at $(x_0,\xi_0) \in X \times \dot{\mathbb{R}}^n$ if there exists a conic neighborhood Γ of (x_0,ξ_0), that is, $\Gamma = \{(y,\eta) : |x_0 - y| < \varepsilon, |\frac{\eta}{|\eta|} - \frac{\xi_0}{|\xi_0|}| < \varepsilon\}$ and a constant C such that*

$$|P(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^m$$

for all $|\xi| \geq C$, $(x,\xi) \in \Gamma$. We say $(x_0,\xi_0) \notin Char(P)$ if it is a non-characteristic point, where $Char(P)$ denotes the characteristic set of P.

Note that the non-characteristic condition only depends on the principal symbol and will not be affected by modulo lower order things in $S^{m-1}_{1,0}$.

Example 3.89. *Suppose the symbol of P satisfies $P(x,\lambda\xi) = \lambda^m P(x,\xi)$ for $\lambda > 0$. Then P is non-characteristic at (x_0,ξ_0) if and only if $P(x_0,\xi_0) \neq 0$.*

Definition 3.90. *We say $P \in \Psi^m_{1,0}(X)$ is elliptic at $x_0 \in X$ if P is non-characteristic at (x_0,ξ_0) for all $\xi_0 \in \dot{\mathbb{R}}^n$. And we say $P \in \Psi^m_{1,0}(X)$ is elliptic on X if P is elliptic at all points of X. That is, if $Char(P) = \varnothing$, then P is elliptic.*

Example 3.91. *The warhorse example for an elliptic operator is $P = -\Delta$, where $P(x,\xi) = |\xi|^2$.*

Example 3.92 (Non-example for elliptic operator)**.** *Let $P = \partial_{x_1} - \Delta_{x'} \in \Psi^2_{1,0}(X)$, then the symbol is $P(x,\xi) = i\xi_1 + |\xi'|^2$. Note that the power of ξ_1 is 1, so we could not expect $|i\xi_1 + |\xi'|^2| \geq \frac{1}{C}(\xi_1^2 + |\xi'|^2)$ to hold for $|\xi| \geq C$. And the principal symbol here is $[\sigma_P] = |\xi'|^2$.*

Example 3.93. *In general, suppose $(a_{ij}) \gg cI$, then the differential operator*

$$P = -\sum \partial_{x_i}\left(a_{ij}\partial_{x_j}\right) + \sum b_j\partial_{x_j} + c$$

with smooth coefficients is elliptic.

The following theorem works for general (ρ,δ), but we only show the case $(\rho,\delta) = (1,0)$. It tells use modulo smoothing operators, elliptic operators are invertible.

Theorem 3.94. *Suppose $P \in \Psi^m_{1,0}(X)$ is elliptic on X. Then there exists properly supported $Q \in \Psi^{-m}_{1,0}(X)$ such that $P \circ Q - I \in \Psi^{-\infty}(X)$, $Q \circ P - I \in \Psi^{-\infty}(X)$ and Q is unique modulo $\Psi^{-\infty}(X)$.*

Proof. We know that for every compact set $K \Subset X$, there exists C_K such that $|P(x,\xi)| \geq \frac{1}{C_K}|\xi|^m$ for $x \in K$, $|\xi| \geq C_K$. (Ellipticity implies this holds at every point, and then holds in a neighborhood, then holds for compact sets by selecting a finite open covering.)

Then we form a locally finite partition of unity $\{\psi_i\} \subset C_c^\infty(X)$ such that $\sum_{i=1}^\infty \psi_i = 1$ on X. Now let

$$Q_0(x,\xi) = \frac{1}{P(x,\xi)} \sum_{i=1}^\infty \psi_i(x) \chi_{\{|\xi| \geq C_{\mathrm{supp}\psi_i}\}}(\xi),$$

where $\chi_A \in C^\infty(\dot{\mathbb{R}}^n)$ satisfies $\chi_A \equiv 1$ on $A \subset \mathbb{R}^n$ and $0 \leq \chi_A \leq 1$. Then $Q_0 \in C^\infty(X \times \mathbb{R}^n)$ and for every compact set $K \Subset X$, there exists $\widetilde{C_K} := \max_{K \cap \mathrm{supp}\psi_i \neq \varnothing} C_{\mathrm{supp}\psi_i}$ such that $Q_0(x,\xi) = \frac{1}{P(x,\xi)}$ for $x \in K, |\xi| \geq \widetilde{C_K}$. Here $\widetilde{C_K}$ is well-defined since the partition is locally finite.

Moreover, for all compact sets $K \Subset X$, we differentiate $PQ_0 = 1$ on $K \times \{|\xi| \geq \widetilde{C_K}\}$ and use induction to prove that $Q_0 \in S^{-m}_{1,0}(X)$ thanks to ellipticity.

By the composition formula (3.19), we have $P\sharp Q_0 = PQ_0$ modulo $S^{-1}_{1,0}$. Moreover, since $1 - PQ_0 \in S^{-\infty}$, we have

$$P\sharp Q_0 = 1 - R, \quad R \in S^{-1}_{1,0},$$
$$Q_0 \sharp P = 1 - T, \quad T \in S^{-1}_{1,0}.$$

Let

$$Q_R = Q_0 \sharp (1 + R + R\sharp R + R\sharp R\sharp R + \cdots) \text{ modulo } S^{-\infty},$$
$$Q_L = (1 + R + R\sharp R + R\sharp R\sharp R + \cdots)\sharp Q_0 \text{ modulo } S^{-\infty},$$

where we use the asymptotic sum in Theorem 3.21. Then

$$P\sharp Q_R = (1-R)\sharp(1 + R + R\sharp R + R\sharp R\sharp R + \cdots) = 1 \text{ modulo } S^{-\infty},$$
$$Q_L \sharp P = (1 + R + R\sharp R + R\sharp R\sharp R + \cdots)\sharp(1-R) = 1 \text{ modulo } S^{-\infty},$$

So $Q_L = Q_L\sharp(P\sharp Q_R) = Q_R$ modulo $S^{-\infty}$. Take $Q(x, D_x)$ be a properly supported pseudodifferential operator such that pseudodifferential operator with the symbol $Q_L(x,\xi) \equiv Q_R(x,\xi)$ modulo $S^{-\infty}$ by introducing a cut-off χ as in Lemma 3.77. Then $P \circ Q - I$ and $Q \circ P - I$ are both smoothing, which completes the proof.

For the uniqueness part, suppose there are two operators Q_1, Q_2 as desired. Then

$$0 \equiv Q_2 \circ (P \circ Q_1 - I) \equiv (Q_2 \circ P - I) \circ Q_1 + Q_1 - Q_2 \equiv Q_1 - Q_2$$

modulo $\Psi^{-\infty}(X)$, which completes the proof. What is essential here is that $\Psi^{-\infty}$ is an ideal in properly supported operators, that is, if $R \in \Psi^{-\infty}$, A properly supported, then $AR \in \Psi^{-\infty}$, $RA \in \Psi^{-\infty}$ thanks to the mapping property of properly supported pseudodifferential operators, (3.15). ∎

Corollary 3.95. *Suppose $P \in \Psi^m_{1,0}(X)$ is elliptic and properly supported on X, then $P : \mathcal{D}'(X)/C^\infty(X) \to \mathcal{D}'(X)/C^\infty(X)$ is an isomorphism.*

Proof. Choose a properly supported Q as in Theorem 3.94. Note that both the definition of properly supported functions and the definition of smoothing operators, Definition 3.70 and Definition 3.65, are symmetric in x and y. Hence, we know ${}^tP$, ${}^tQ$ are also properly supported, hence ${}^tQ \circ {}^tP - I = {}^t(P \circ Q - I)$ is continuous $\mathcal{E}'(X) \to \mathcal{E}'(X)$ and ${}^t(P \circ Q - I)$ is still smoothing. Combined the two properties above, ${}^t(P \circ Q - I) : \mathcal{E}'(X) \to C_c^\infty(X)$. Thus, by taking the adjoint, we know $P \circ Q - I : \mathcal{D}'(X) \to C^\infty(X)$. Similarly, we have $Q \circ P - I : \mathcal{D}'(X) \to C^\infty(X)$. Now we complete the proof. □

Corollary 3.96. *Suppose $P \in \Psi_{1,0}^m(X)$ is elliptic and properly supported on X. Then singsupp $Pu =$ singsupp u for $u \in \mathcal{E}'(X)$.*

Proof. From the semilocal property Theorem 3.64, singsupp $Pu \subset$ singsupp u. On the other hand, singsupp $Q \circ Pu \subset$ singsupp Pu. Since $(Q \circ P - I)u \in C^\infty(X)$, we know singsupp $Q \circ Pu =$ singsupp u, which completes the proof. □

3.8. **Mapping properties of pseudodifferential operators between $H^s(\mathbb{R}^n)$.** In order to show a locally solvability theorem for differential operators with smooth coefficients, Theorem 3.107, we need to learn the mapping property on Sobolev spaces.

Definition 3.97. *We define*

$$\overline{S}_{\rho,\delta}^m = \left\{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha,\beta}, |\partial_x^\alpha \partial_\theta^\beta a| \le C_{\alpha,\beta} \langle \theta \rangle^{m - \rho|\beta| + \delta|\alpha|} \right\}.$$

Note that all the previous properties we had before are true for this new class. We get a new class of operators:

Definition 3.98. *We denote $\overline{\Psi}_{\rho,\delta}^m(\mathbb{R}^n)$ for all the operators $A : \mathscr{S} \to \mathscr{S}$ such that*

$$Au(x) = \frac{1}{(2\pi)^n} \int a(x,\xi) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi,$$

understood as an oscillatory integral for $u \in C_c^\infty(\mathbb{R}^n)$ or understood as an iterated integral, then it also makes sense for $u \in \mathscr{S}(\mathbb{R}^n)$. Here $a \in \overline{S}_{\rho,\delta}^m$.

And we had a fact that is a little better than the result in Theorem 3.76.

Theorem 3.99. *Suppose $A \in \overline{\Psi}_{\rho,\delta}^m(\mathbb{R}^n)$, then $A = A_1 + A_2$, where $A_1 \in \overline{\Psi}_{\rho,\delta}^m(\mathbb{R}^n)$ is properly supported and $K_{A_2} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies*

$$|\partial_x^\alpha \partial_y^\beta K_{A_2}(x,y)| \le C_{\alpha,\beta,N} \langle x - y \rangle^{-N},$$

that is, it has decay properties away from the diagonal.

Proof. We choose χ as in Lemma 3.77, then we compute

$$\begin{aligned}
K_{A_2}(x,y) &= (1 - \chi(x,y)) \int e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi \\
&= (1 - \chi(x,y)) \int \left(\frac{(x-y)\cdot D_\xi}{|x-y|^2} \right)^N e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi \\
&= \frac{1 - \chi(x,y)}{|x-y|^{2N}} \int e^{i(x-y)\cdot\xi} \left((x-y)\cdot D_\xi\right)^N a(x,\xi)\, d\xi,
\end{aligned}$$

where the first integral is understood as an oscillatory integral and then we can do integration by parts in the second and third equality. Though in the definition of the oscillatory integral, we should pair it with a test function $u(y)$ and then integration by parts is valid, we note that the differential operator used here to integrate by parts only has D_ξ, so the same type of argument in the equation above also holds if we pair it with $u(y)$, we omit it for convenience.

Note that $|D_\xi^N a| \leq C_N \langle\xi\rangle^{m-\rho N}$ for all N, so the integration on the right hand side of the equation above is smooth in x and y and have the bound $|K_{A_2}(x,y)| \leq \widetilde{C_N}\langle x-y\rangle^{-N}$. Obviously, similar estimates hold for derivatives. $\square$

Actually, A_2 in the above theorem has good mapping properties. It takes bad things like H^{-M} to very good things like H^M for $M \geq 0$, which will be proved in Theorem 3.101. Before showing this, we need a lemma named after Schur.

Lemma 3.100 (Schur's lemma)**.** *Suppose $K_B(x,y)$ is the kernel of B and*

$$\sup_x \int |K_B(x,y)|\,dy,\ \sup_y \int |K_B(x,y)|\,dx \leq C$$

Then $\|B\|_{L^2\to L^2} \leq C$.

Proof. The proof is a direct computation as follows.

$$\begin{aligned}\|Bu\|_2^2 = &\int \left|\int K_B(x,y)u(y)\,dy\right|^2 dx \leq \int \left(\int |K_B(x,y)|\,dy \int |K_B(x,y)|u(y)^2\,dy\right) dx \\ \leq &C \iint |K_B(x,y)|u(y)^2\,dy\,dx \leq C^2 \int u(y)^2\,dy.\end{aligned}$$

$\square$

This lemma is nontrivial even for the case of matrices.

Theorem 3.101. *Suppose $K_A(x,y)$ is the kernel of A and*

$$|\partial_x^\alpha \partial_y^\beta K_A(x,y)| \leq C_{\alpha,\beta,N}\langle x-y\rangle^{-N}$$

for all α, β, N. Then $A: H^r(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s, r \in \mathbb{R}$.

Proof. Note that it suffices to prove $A: H^{-M}(\mathbb{R}^n) \to H^M(\mathbb{R}^n)$ for all $M \in \mathbb{N}$. Thanks to Schur's lemma, this is true for $M = 0$. The kernels $\partial_x^\alpha K_A(x,y)$ satisfy the assumptions for the kernel in Schur's lemma and it is easy to verify that $\partial_x^\alpha K_A(x,y)$ is the kernel for $\partial_x^\alpha A$, so $A: L^2(\mathbb{R}^n) \to H^M(\mathbb{R}^n)$ for all $M \in \mathbb{N}$. For $u \in H^{-M}(\mathbb{R}^n)$, we know that $\widehat{u}(\xi)\langle\xi\rangle^{-M} \in L^2$ and there exists $v \in L^2$ such that $v = \mathcal{F}^{-1}\left(\widehat{u}(\xi)\langle\xi\rangle^{-M}\right)$, $u = |\nabla|^M v$. Note that for all x, $K_A(x,\cdot) \in H^M(\mathbb{R}^n)$, so $\int K_A(x,y)u(y)\,dy$ can be understood as the distributional pairing

$$\langle u(y), K_A(x,y)\rangle = (-1)^{Mn}\langle v(y), |\nabla|_y^M K_A(x,y)\rangle.$$

Since the kernel $|\nabla|_y^M K_A(x,y)$ also satisfies the assumptions for the kernel in Schur's lemma, we know $A: H^{-M}(\mathbb{R}^n) \to H^M(\mathbb{R}^n)$ for all $M \in \mathbb{N}$, which completes the proof. $\square$

Now we denote $\overline{\Psi}^m(\mathbb{R}^n) = \overline{\Psi}_{1,0}^m(\mathbb{R}^n)$ for simplicity.

Theorem 3.102. *Suppose $A \in \overline{\Psi}^0_{1,0}(\mathbb{R}^n)$, then $A : L^2 \to L^2$.*

Proof. From Theorem 3.99 and Theorem 3.101, without loss of generality, we can assume A is properly supported in each step below.

Step 1: First, we assume that $A \in \overline{\Psi}^{-n-1}(\mathbb{R}^n)$. We will claim that $A : L^2 \to L^2$. In fact, the kernel K_A satisfies

$$|K_A(x,y)| = \frac{1}{(2\pi)^n}\left|\int a(x,\xi)e^{i(x-y)\cdot\xi}\,d\xi\right| \le \frac{1}{(2\pi)^n}\int |a(x,\xi)|\,d\xi \le C < \infty.$$

Since $\partial_\xi^\alpha a(x,\xi) \in \overline{S}^{-n-1-|\alpha|}$ for all α,

$$|(x-y)^\alpha K_A(x,y)| = \frac{1}{(2\pi)^n}\left|\int i^{|\alpha|}\partial_\xi^\alpha a(x,\xi)e^{i(x-y)\cdot\xi}\,d\xi\right| \le \frac{1}{(2\pi)^n}\int |\partial_\xi^\alpha a(x,\xi)|\,d\xi \le C_\alpha < \infty.$$

Combining the two estimates above, we have

$$|K_A(x,y)| \le C_N\langle x-y\rangle^{-N}$$

for all $N > 0$, then apply Schur's lemma, we know $A : L^2 \to L^2$.

Step 2: For $A \in \overline{\Psi}^{-k}(\mathbb{R}^n)$, $k = 1, 2, \cdots$, then

$$\|Au\|_2^2 = \langle Au, Au\rangle = \langle A^*Au, u\rangle,$$

where $A^*A \in \overline{\Psi}^{-2k}(\mathbb{R}^n)$. Since for $0 < l \le n+1$, $2l \ge l+1$, then by a finite induction, if we start with $l = n+1$, we know for all $k = 1, 2, \cdots$,,

$$\|Au\|_2^2 = \langle A^*Au, u\rangle \le C\|u\|_2^2,$$

that is, $A : L^2 \to L^2$.

Step 3: For $A \in \overline{\Psi}^0(\mathbb{R}^n)$, choose $M > 2\sup|a(x,\xi)|^2$, we claim that

$$c(x,\xi) := (M - |a(x,\xi)|^2)^{\frac{1}{2}} \in \overline{S}^0.$$

Since $M - |a(x,\xi)|^2 > \frac{1}{2}M$, we know $c \in C^\infty$. Moreover,

$$\left|\partial_x^\alpha\partial_\xi^\beta\left((M-|a(x,\xi)|^2)^{\frac{1}{2}}\right)\right| \le \left|\frac{\sum_{|\alpha_1|+\cdots+|\alpha_p|=|\alpha|,|\beta_1|+\cdots+|\beta_p|=|\beta|}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}(a(x,\xi)^2)\cdots\partial_x^{\alpha_p}\partial_\xi^{\beta_p}(a(x,\xi)^2)}{(M-|a(x,\xi)|^2)^{\frac{|\alpha|+|\beta|}{2}}}\right|$$

$$\le \left(\frac{2}{M}\right)^{\frac{|\alpha|+|\beta|}{2}}\sum_{|\alpha_1|+\cdots+|\alpha_p|=|\alpha|,|\beta_1|+\cdots+|\beta_p|=|\beta|} C\langle\xi\rangle^{-|\beta_1|-\cdots-|\beta_p|} \le C\langle\xi\rangle^{-|\beta|},$$

which implies the claim is true.

Now we look at the operator $c(x,D)^*c(x,D)$. By the expansion of the adjoint, we know $c(x,D)^* = c(x,D) + e(x,D)$, where $e \in \overline{S}^{-1}$. Hence,

$$c(x,D)^*c(x,D) = c(x,D)^2 + c(x,D)e(x,D) = c^2(x,D) + f(x,D) + c(x,D)e(x,D),$$

where $c(x,D)^2 = c^2(x,D) + f(x,D)$ with $f \in \overline{S}^{-1}$. So $c(x,D)^*c(x,D) = (M - |a|^2)(x,D) + g(x,D)$, where $g \in \overline{S}^{-1}$. Again, $|a|^2(x,D) = a(x,D)^*a(x,D) + h(x,D)$ with $h \in \overline{S}^{-1}$. Thus, we get

$$c(x,D)^*c(x,D) = M - a(x,D)^*a(x,D) + r(x,D),$$

where $r \in \overline{S}^{-1}$. Take $u \in \mathscr{S}$, then

$$\|a(x,D)u\|_2^2 = \langle a(x,D)^* a(x,D)u, u\rangle = M\|u\|_2^2 - \langle c(x,D)^* c(x,D)u, u\rangle + \langle r(x,D)u, u\rangle$$
$$= M\|u\|_2^2 - \|c(x,D)u\|_2^2 + \langle r(x,D)u, u\rangle \leq M\|u\|_2^2 + \langle r(x,D)u, u\rangle.$$

And by the previous step, we know $\langle r(x,D)u, u\rangle \leq C\|u\|_2^2$. Hence, we have shown that $a(x,D)$ is bounded on L^2.

□

Remark 3.103. This argument holds for $\rho > \delta$. When $\rho = \delta$, the result is still true, which is called the Calderon-Vaillancourt theorem, but we need to use a different proof.

Theorem 3.104. *Suppose $A \in \overline{\Psi}^m(\mathbb{R}^n)$, then for all $s \in \mathbb{R}$, we have the mapping property*

$$A : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n).$$

Proof. We define the operator $\Lambda_r := (I - \Delta)^{\frac{r}{2}}$ as a Fourier multiplier, then $\|\Lambda_r u\|_{L^2} = \|u\|_{H^r}$, that is, $\Lambda_r : H^r \to L^2$ is an isomorphism. Then it suffices to prove

$$\|\Lambda_s A u\|_{L^2} \leq C\|\Lambda_{s+m} u\|_{L^2},$$

for all $u \in H^{s+m}(\mathbb{R}^n)$, and this is equivalent to

$$\|\Lambda_s A \Lambda_{-s-m} w\|_{L^2} \leq C\|w\|_{L^2},$$

for all $w \in L^2$. By density, it suffices to show

$$\|\Lambda_s A \Lambda_{-s-m} w\|_{L^2} \leq C\|w\|_{L^2},$$

for all $w \in \mathscr{S}$.

From Theorem 3.99 and Theorem 3.101, without loss of generality, we can assume A is properly supported, then the composition formula holds for A. Moreover, $\Lambda_s \in \overline{\Psi}^s$, $A \in \overline{\Psi}^m$ and $\Lambda_{-s-m} \in \overline{\Psi}^{-s-m}$, so $\Lambda_s A \Lambda_{-s-m} \in \overline{\Psi}^0$, which implies $\Lambda_s A \Lambda_{-s-m} : L^2 \to L^2$, which completes the proof. □

Now, we go back to the original space with an open set X.

We denote $H^s_{comp} = \mathcal{E}' \cap H^s_{loc}$ be the spaces of compactly supported distributions in H^s.

Theorem 3.105. *Suppose $A \in \Psi^m(X)$, then for all $s \in \mathbb{R}$, we have the mapping property*

$$A : H^{s+m}_{comp}(X) \to H^s_{loc}(X).$$

Proof. We just need to notice that since for all $\psi \in C_c^\infty(X)$, $\varphi \in C_c^\infty(X)$ such that $\varphi \equiv 1$ on supp$u$, we have $\psi A \varphi \in \overline{\Psi}^m(\mathbb{R}^n)$ due to the fact that we already localize x, y in a compact set, so it is uniform. Then we can apply Theorem 3.104 to conclude. □

Theorem 3.106. *Suppose $A \in \Psi^m(X)$ is properly supported, then for all $s \in \mathbb{R}$, we have the mapping property*

$$A : H^{s+m}_{comp}(X) \to H^s_{comp}(X), \quad A : H^{s+m}_{loc}(X) \to H^s_{loc}(X).$$

Proof. The first follows directly from the preceding theorem and the property $A : \mathcal{E}'(X) \to \mathcal{E}'(X)$ for properly supported operators. For the second one, we need to use the definition of proper support by denoting $C = \text{supp} K_A$. For any $\psi \in C_c^\infty(X)$, we choose $\varphi \in C_c^\infty$ such that it is 1 on $C^{-1}(\text{supp}\psi)$, which is a compact set. So $\psi A = \psi A \varphi$, which implies the result by composing $\varphi : H^{s+m}_{loc}(X) \to H^{s+m}_{comp}(X)$, $\psi A : H^{s+m}_{comp}(X) \to H^s_{comp}(X)$ with support depending on ψ, so it is in $H^s_{loc}(X)$. ☐

3.9. **Local solvability of elliptic differential operators.**

Theorem 3.107 (Local solvability). *Suppose P is an elliptic differential operator with smooth coefficients, that is, $P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha$ with $a_\alpha \in C^\infty(X)$. Then for every $x_0 \in X$, there exists an open neighborhood $V \subset X$ such that for all $f \in C^\infty(V)$ (or $f \in \mathcal{D}'(V)$), for all open set $W \Subset V$, there exists $u \in C^\infty(V)$ (or $u \in \mathcal{D}'(V)$), such that $Pu = f$ in W.*

Proof. **Step 1:** We claim for all compact sets $K \Subset X$, there exists C, such that

$$\|u\|_{H^m} \le C \left(\|P^* u\|_{L^2} + \|u\|_{L^2}\right)$$

for all $u \in \mathcal{E}'(K) \cap H^m(K)$.

From Theorem 3.94, there exists $B \in \Psi^{-m}$ properly supported such that $BP^* = I + K$ and $K \in \Psi^{-\infty}$. Note that the support of K_P lies exactly on the diagonal and K_{B^*} is properly supported, so K_{PB^*} is properly supported. So we know $K \in \Psi^{-\infty}$ is properly supported and hence $B, K : L^2_{comp} \to H^m_{comp}$ satisfies the estimates

$$\|u\|_{H^m} = \|BP^*u - Ku\|_{H^m} \le \|BP^*u\|_{H^m} + \|Ku\|_{H^m} \le C\left(\|P^*u\|_{L^2} + \|u\|_{L^2}\right)$$

for $u \in \mathcal{E}'(K) \cap H^m(K)$.

Step 2: Now we want to upgrade the a priori estimates above. Suppose $V \subset B(x_0, \varepsilon)$ and we consider $u \in \mathcal{E}'(V) \cap H^m(\mathbb{R}^n)$, we claim that $\|u\|_{L^2} \le C\varepsilon^m \|u\|_{H^m}$.

Recall the Poincare inequality gives us that for all $v \in H^m(B(0,1))$, $\text{supp} v \Subset B(0,1)$, we have

$$\|v\|_{L^2} \le C \sum_{|\alpha|=1} \|\partial^\alpha v\|_{L^2} \le C_2 \sum_{|\alpha|=2} \|\partial^\alpha v\|_{L^2} \le \cdots \le C_m \sum_{|\alpha|=m} \|\partial^\alpha v\|_{L^2}.$$

Furthermore, rescaling tells us for all $u \in H^m(B(x_0, \varepsilon))$, $\text{supp} v \Subset B(x_0, \varepsilon)$,

$$\|u\|_{L^2} \le C_m \varepsilon^m \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^2} \le C\varepsilon^m \|u\|_{H^m}.$$

So if ε is small enough, we combine this with the estimates in Step 1,

$$\|\varphi\|_{H^m} \le C\|P^*\varphi\|_{L^2},$$

for all $\varphi \in \mathcal{E}'(V) \cap H^m$.

Step 3: We claim that we can reduce the proof to the case when $f \in C^\infty(V)$. Suppose $\widetilde{B}$ satisfies $P\widetilde{B} = I + K$ such that $\widetilde{B} \in \Psi^{-m}$ and $K \in \Psi^{-\infty}$. For all $v \in \mathcal{D}'(V)$, put $\widetilde{u} = \widetilde{B}v$, then $P\widetilde{u} = P\widetilde{B}v = v + \widetilde{v}$ where $\widetilde{v} = Kv \in C^\infty(V)$. This gives the solvability modulo C^∞ functions. Then the problem is reduced to the case when $v \in C^\infty(V)$.

Step 4: We can assume $v \in C^\infty(V)$, $W \Subset V$, then we define a linear functional l on $H^m \cap \mathcal{E}'(W)$ given by $l(\varphi) = \langle \varphi, v\rangle$. It satisfies the estimate

$$|l(\varphi)| \le C(v, W)\|\varphi\|_{H^m} \le \widetilde{C}\|P^*\varphi\|_{L^2}.$$

Let $L = \{P^*\varphi \in L^2 \cap \mathcal{E}'(W) : \varphi \in H^m \cap \mathcal{E}'(W)\}$, which is a linear space. Then k defined by $k(P^*\varphi) = l(\varphi)$ is a bounded linear functional on L.

By the Hahn-Banach theorem, k has a bounded extension to L^2, that is, $k : L^2 \to \mathbb{C}$, so there exists $u \in L^2$ such that $k(\psi) = \langle \psi, u\rangle$. So

$$\langle \varphi, v\rangle = l(\varphi) = k(P^*\varphi) = \langle P^*\varphi, u\rangle = \langle \varphi, Pu\rangle$$

for all $\varphi \in H^m \cap \mathcal{E}'(W)$. So $Pu = v$ in W.

Step 5: Finally, from Corollary 3.96, we know singsupp Pu = singsupp u, so if $v \in C^\infty(V)$, we know $u \in C^\infty(V)$. □

Remark 3.108. This proof is an example of the duality argument.

3.10. Wavefront sets.

In fact, we have a more general version of Corollary 3.96.

Theorem 3.109. *For $P \in \Psi^m_{1,0}(X)$, we have*

$$\text{singsupp } u \subset \text{singsupp } Pu \cup \pi(Char(P)),$$

where $\pi : X \times \mathbb{R}^n \to X$.

We just provide a sketch of proof here. We construct a local parametrix of P near any non-characteristic point. To be more specific, we construct Q such that $QR = I + R_1$ and $PQ = I + R_2$ with $R_j(x,\xi) = O(\langle\xi\rangle^{-\infty})$ near $(x_0, \xi_0) \notin Char(P)$. The idea of proof is the same as for the elliptic case and we need to introduce some cut-off functions.

Here we can see an example of this result.

Example 3.110. *Let $P = D^2_{x_n} - |D_{x'}|^2$, which corresponds to the wave equation. Suppose $PE_0 = \delta_0(x)$ and we consider the forward solution, then it stays in the cone $\{x_n^2 = |x'|^2, x_n > 0\}$. In this case, $Char(P) = \{(x,\xi) : \xi_n^2 = |\xi'|^2\}$, then $\pi(Char(P)) = \mathbb{R}^n$, so we do not get any interesting information from the theorem above.*

We need to know more information about the wave to determine where the wave lives at some subsequent time. By Huygens Principle, we need to know in which direction the wave is moving. So we introduce a new concept below.

Definition 3.111. *The wavefront set $WF(u) \subset X \times \dot{\mathbb{R}}^n$ of $u \in \mathcal{D}'(X)$ is defined by*

$$WF(u) := \bigcap_{P \in \Psi^\infty, Pu \in C^\infty} Char(P).$$

Intuitively, we are looking at all possible places where the symbol, roughly speaking, vanishes. Note that if $Char(P) = \varnothing$, then $Pu \in C^\infty$ implies $u \in C^\infty$.

Example 3.112. *We claim*

$$WF(\delta_0) = \{(0,\xi) : \xi \in \dot{\mathbb{R}}^N\}.$$

Note that

$$P(x,D)\delta_0(x) = \frac{1}{(2\pi)^n}\int p(x,\xi)e^{i(x-y)\cdot\xi}\delta_0(y)\,dy\,d\xi = \frac{1}{(2\pi)^n}\int p(x,\xi)e^{ix\cdot\xi}\,d\xi,$$

so if $p(x,\xi) = p(x)$, then $P(x,D)\delta_0(x) = p(0)\delta_0(x) \notin C^\infty$, if $p(0) \neq 0$. Now we can imagine $WF(\delta_0) = \{(0,\xi) : \xi \in \dot{\mathbb{R}}^N\}$. To prove this rigorously, we need the following characterization theorem for wavefront set.

Theorem 3.113. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $(x_0, \xi_0) \in \mathbb{R}^n \times \dot{\mathbb{R}}^n$. Then $(x_0, \xi_0) \notin WF(u)$ if and only if there exists $\phi \in C_c^\infty(\mathbb{R}^n), \phi(x_0) \neq 0$ and $\varepsilon > 0$ such that $|\widehat{\phi u}(\xi)| \leq C_N |\xi|^{-N}$ for $\xi \in \mathbb{R}^n, |\xi| \geq 1$, $\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \varepsilon$.*

Proof. See [5, Proposition 6.19] or [4, Proposition 7.4] for a proof. □

We also have

$$\text{singsupp } u = \pi(WF(u)).$$

See [4, Proposition 7.3] for a proof.

3.11. **Parametrix construction for hyperbolic equations.** One case for which we can construct a parametrix for P (non-elliptic) are strictly hyperbolic operators.

Definition 3.114. *Let $P = P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ and $p(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is the principal symbol. We say that P is strictly hyperbolic with respect to the family of hyperplanes $x_n = const$ if for every $\xi' \neq 0$, $p(x, \xi) = 0$ has exactly m distinct real roots $\xi_n = \lambda_j(x, \xi')$ for $j = 1, \cdots, m$.*

Example 3.115. *Let $P = D_{x_n}^2 - |D_{x'}|^2$, then $\xi_n = \pm|\xi'|$ is two distinct roots for $p = 0$ if $\xi' \neq 0$. Thus, P is strictly hyperbolic.*

If P is strictly hyperbolic, then there exists $f(x) \neq 0$ such that

$$p(x, \xi) = f(x) \prod_{j=1}^{m} (\xi_n - \lambda_j(x, \xi')) .$$

When $f \neq 1$, we have

$$P(x, \xi) = D_{x_n}^m + A_1(x, D_{x'}) D_{x_n}^{m-1} + \cdots + A_m(x, D_{x'}).$$

Here is an idea of parametrix construction. For each root $\lambda = \lambda_\nu$, we shall find a certain operator $E_\nu : \mathcal{E}'(\omega) \to \mathcal{D}'(X)$ such that the kernel $P \circ E_\nu \in C^\infty$. We write

$$E_\nu(x, y') = \frac{1}{(2\pi)^{n-1}} \int e^{i\varphi_\nu(x,\eta') - iy'\cdot\eta'} a_\nu(x, \eta')\, d\eta',$$

with a suitable phase function φ_ν and $a_\nu \in S^0$, so φ_ν is homogeneous of degree 1 in η'. If $\varphi_\nu(0, x', \eta') = x' \cdot \eta'$ and $a_\nu(0, x', \eta') = 1$, then $E_\nu(0, x', y') = \delta(x' - y')$. Since $(e^{-i\varphi_\nu} D_x e^{i\varphi_\nu})\, u = (D_x + \partial_x \varphi) u$, we have

$$P(e^{i\varphi_\nu(x,\eta')} a_\nu(x, \eta')) = e^{i\varphi_\nu(x,\eta')} b_\nu(x, \eta'),$$

where

$$b_\nu(x, \eta') = \left(\sum_{|\alpha| \leq m} a_\alpha(x)(D_x + \partial_x \varphi)^\alpha \right) a_\nu(x, \eta').$$

Since φ_ν is homogeneous of degree 1 in η', we know φ is S^1 by Example 3.11, thus $b_\nu(x, \eta') \in S^m$. However, we want $b_\nu \in S^{-\infty}$.

Here we shall use the WKB method, first noted by Peter Lax in 1950s for hyperbolic equations. Modulo S^{m-1}, we have

$$b_\nu(x,\eta') \equiv \left(\sum_{|\alpha|=m} a_\alpha(x)(\partial_x\varphi)^\alpha\right) a_\nu(x,\eta') \equiv p(x,\partial_x\varphi)a_\nu(x,\eta'),$$

where $p(x,\xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. In order to find a non-vanishing a with $b\in S^{-\infty}$, we require that $p(x,\partial_x\varphi_\nu)=0$. So, it suffices to choose $\varphi_n u$ to be the solution of $q_\nu(x,\partial_x\varphi_\nu)=0$ with the initial value $\varphi_\nu(0,x',\eta')=x'\cdot\eta'$. where $q_\nu := \xi_n - \lambda_\nu(x,\xi')$. We can use the method of characteristics to get a local solution φ_ν. (Though the method of characteristics can give smooth solutions, shocks can form so the solutions are only local.) Then we extend the solution on $X\times S^{n-2}$ to a smooth solution on $X\times\dot{\mathbb{R}}^{n-1}$ by homogeneity of degree 1. Hence, $b\in S^{m-1}$.

Modulo S^{m-2}, we have

$$b \equiv \sum_{j=1}^{n} p_{\xi_j}(x,\partial_x\varphi_\nu)D_{x_j}a_\nu + f(x,\eta')a_\nu,$$

where $p_{\xi_j}, f\in S^{m-1}$. Denote $L(a_\nu) := \sum_{j=1}^n p_{\xi_j}(x,\partial_x\varphi_\nu)D_{x_j}a_\nu + f(x,\eta')a_\nu$, then we want to express a_ν as an asymptotic expansion $a_\nu \sim a_{0,\nu}+a_{1,\nu}+\cdots$, where $a_{j,\nu}\in S^{-j}$. Then we solve the first order linear PDE $L(a_{0,\nu})=0$ by the method of characteristics and then we will iteratively get a sequence first order linear PDE. Finally, we shall get the desired a.

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