THE CAUCHY PROBLEM FOR WAVE MAPS

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Contents

1. Wave maps and main results

Consider wave maps $u : (\mathbb{R}^{m+1}, \eta) \to (N, h)$ with target manifold N^k being a complete Riemannian manifold without boundary,

$$D^\alpha \partial_\alpha u = 0, \tag{1.1}$$

where we raise and lower indices with the Minkowski metric η. Moreover, D is the covariant derivative in the pullback bundle u^*TN.

As noted in [10], by noting the special solutions contained on geodesics, one easily sees that N needs to be geodesically complete if one wants to consider the existence of global

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Date: September 29, 2023.

This is a note for an expository talk mainly on [7]. We also make a brief introduction on some aspects covered in [8] and [5] in the last section.

solution. Moreover, as we will see, for the critical scaling case we would focus on, small data local existence will lead to small data global existence in view of scaling.

We make a further assumption that N is parallelizable, that is, there exist smooth vector fields $\bar{e}_1, \cdots, \bar{e}_k$ such that at each $p \in N$ the collection $\bar{e}_1(p), \cdots, \bar{e}_k(p)$ is an orthonormal basis for T_pN. This condition is natural, see [2, Section 4], in which they proved that if N is compact, then we can find an embedding $(N, h) \hookrightarrow (Z, \tilde{h})$ such that Z is parallelizable and $\tilde{h} = h$ on $i(N)$. The proof is based on the Whitney embedding theorem, normal exponential map and a partition of unity argument. (For example, S^2 is not parallelizable by the hairy ball theorem but $\mathbb{R}^3$ is trivially parallelizable.) Also, we know $\mathbb{S}^3$ is parallelizable.

In local coordinates, we can express the intrinsic formulation by

$$\Box u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0,$$

where Γ^a_{bc}'s are the Christoffel symbols of the target N. To compare, we also have an extrinsic formulation of our wave maps. By using the Nash embedding theorem, we regard N as a submanifold of some Euclidean space $\mathbb{R}^n$, namely,

$$u = (u^1, \cdots, u^n) : \mathbb{R}^{m+1} \to N \hookrightarrow \mathbb{R}^n.$$

Then (1.1) takes the form

$$u^i_{tt} - \Delta u^i = B^i_{jk}(\partial_\alpha u^j, \partial^\alpha u^k), \quad 1 \leq i \leq n, \tag{1.2}$$

where $B(p) : T_pN \times T_pN \to (T_pN)^\perp$ is the second fundamental form of $N \subset \mathbb{R}^n$ at any $p \in N$. The derivation can be found in [9].

From the intrinsic formulation, one can see that the equation is invariant under the scaling $\phi \mapsto \phi(\lambda t, \lambda x)$, which tells us the scale-invariant case is $\dot{H}^{m/2} \times \dot{H}^{m/2-1}$. The small data global wellposedness in this critical regularity case is studied in [8] and [7].

Before we dive into the discussion of this case, we begin with a local existence result in the simplest setting :

$$\begin{aligned} H^{s+1}_c(\mathbb{R}^m; TN) = \big\{ &(u_0, u_1) \in L^2_{\text{loc}}(\mathbb{R}^m; TN) : \\ &u_0 \in H^{s+1}\left(\mathbb{R}^m; \mathbb{R}^k\right), u_1 \in H^s\left(\mathbb{R}^m; \mathbb{R}^k\right), \operatorname{supp}\left(u_1, \nabla u_0\right) \subset\subset \mathbb{R}^m \big\}, \end{aligned} \tag{1.3}$$

where $s > m/2$. We consider in the extrinsic formulation (1.2) and view the right hand side as our source term

$$f := B^i_{jk}(\partial_\alpha u^j, \partial^\alpha u^k)$$

under the assumption that second fundamental form is bounded. By energy estimates for the wave equation, we have

$$\frac{d}{dt}\|\partial^{s+1}u(t)\|_{L^2} \lesssim \|\partial^s f\|_{L^2}.$$

Thanks to the Sobolev embedding $H^s \subset C^0 \cap L^\infty$ and the Gagliardo-Nirenberg interpolation, we are able to prove

$$\|\partial^s\left(A(u)\left(\partial_\alpha u, \partial^\alpha u\right)\right)\|_{L^2} \lesssim \left(1 + \left\|\partial^{s+1} u\right\|_{L^2}\right)$$

uniformly in $0 \leq t \leq T$ when $\sup_{0 \leq t \leq T} \|\partial^{s+1}u(t)\|_{L^2} \leq C_0$. Then we can obtain bounds for $\|\partial^{s+1}u(t)\|_{L^\infty L^2}$, which is the usual first step for a proof of local existence of nonlinear wave equation by using a sequence of iterations.

Therefore, we are able to obtain local wellposedness result for initial data in $H_c^{s+1}(M;TN)$ with $s > m/2$. In particular, if $(u_0, u_1) \in C^\infty(\mathbb{R}^m; TN)$ with $\operatorname{supp}(u_1, \nabla u_0) \subset\subset \mathbb{R}^m$, then the solution obtained is also smooth. See [1] for further discussion.

1.1. **Main results.** The main purpose of this note is to consider the Cauchy problem for wave maps with initial data

$$(u, u_t)|_{t=0} = (u_0, u_1) \in H^{m/2} \times H^{m/2-1}(\mathbb{R}^m; TN), \tag{1.4}$$

where u_0 takes value in N and u_1 takes value in $T_{u_0}N$. However, one cannot even expect $u_0 \in L^2$ if u_0 takes values on the sphere. To get around this, we abuse the notation and assume constant functions are in H^s with zero norm. See [8, Section 1, footnote].

The main result we are going to discuss is as follows :

Theorem 1.1 (Shatah and Struwe, [7]). *Suppose N satisfies the conditions stated before and has bounded curvature in the sense that the curvature operator R and the second fundamental form B and all their derivatives are bounded, and let $m \geq 4$. Then there is a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1) \in H^{m/2} \times H^{m/2-1}(\mathbb{R}^m; TN)$ satisfying*

$$\|u_0\|_{\dot{H}^{m/2}} + \|u_1\|_{\dot{H}^{m/2-1}} < \varepsilon_0,$$

there exists a unique global solution $u \in C^0(\mathbb{R}; H^{m/2}) \cap C^1(\mathbb{R}; H^{m/2-1})$ of (1.1) and (1.4) satisfying

$$\sup_t \|du(t)\|_{\dot{H}^{m/2-1}} + \int_{\mathbb{R}} \|du(t)\|^2_{L^{2m}(\mathbb{R}^m)}\, dt \leq C\varepsilon_0$$

and preserving any higher regularity of the data.

We will notice that the boundedness of B is used to derive the equivalence of norms.

1.2. **Difficulties for critical scaling : a failure of direct application of Strichartz.** We consider the target $N = \mathbb{S}^k$, then the wave maps equation can be written in the extrinsic form

$$\Box\phi = \phi\partial^\mu\phi\partial_\mu\phi. \tag{1.5}$$

It is known that the cubic covariant scalar wave equation is locally wellposed in the scale invariant space for higher dimensions by Strichartz estimates (see [9, Page 304, Exercise 6.33]). It may seem natural to think of applying Strichartz estimates to (1.5) as well.

For (1.5), we write

$$\Box\partial^{m/2-1}\phi = \phi\partial^\mu\phi\partial_\mu\partial^{m/2-1}\phi + \cdots .$$

In order to apply the Strichartz estimates

$$\||D|^s\phi\|_{L^qL^r} \lesssim \|\vec{\phi}(0)\|_{\dot{H}^1\times L^2} + \||D|^{s'}\Box\phi\|_{L^{q'}L^{r'}},$$

where

$$\frac{1}{q} + \frac{m}{r} - s = \frac{m}{2} - 1 = \frac{1}{q'} + \frac{m}{r'} - s' - 2.$$

Therefore, we have some special endpoints :

$$s = 1, (q, r) = (\infty, 2), \quad s' = 0, (q', r') = (1, 2).$$

For any other admissible pair (q,r), we can only achieve $s<1$, so in order not to lose any derivative, we need to use $L^\infty L^2$ on the left hand side. Similarly, in order not to gain any derivative on the right hand side, we need to put the nonlinearity in L^1L^2, so we obtain

$$\left\|\phi\partial^\mu\phi\partial_\mu\partial^{m/2-1}\phi\right\|_{L^1L^2}\lesssim\|\partial_\mu\partial^{m/2-1}\phi\|_{L^\infty L^2}\|\phi\partial^\mu\phi\|_{L^1L^\infty}.$$

However, the Strichartz estimates when $r=\infty$ is subtle since there are no summability and equivalence of norms using the square functions.

In the critical scaling case, we need to exploit the geometric structure. [8] uses microlocal gauge constructed using paradifferential calculus, which enable to treat error terms in L^1L^2 and apply Strichartz estimates to conclude. See Section 4.1. Coulomb gauge is used in the work [7] that we are going to dicuss in detail, where the existence is proved in [12] using a continuity method in which they essentially use a perturbative method and elliptic estimates. For lower dimensions, things become more complicated. One may need to use the null structure as well. See Section 4.2

2. Preliminaries on Lorentz spaces, interpolations, Strichartz estimates

For this section, we refer to [4], [11] and [3].

2.1. Lorentz spaces.

For $1\le p<\infty$ and $f:\mathbb{R}^d\to\mathbb{C}$ we define

$$\|f\|^*_{L^p_{\text{weak}}\left(\mathbb{R}^d\right)}:=\sup_{\lambda>0}\lambda|\{x:|f(x)|>\lambda\}|^{1/p} \tag{2.1}$$

and the weak L^p space

$$L^p_{\text{weak}}\left(\mathbb{R}^d\right):=\left\{f:\|f\|^*_{L^p_{\text{weak}}}\left(\mathbb{R}^d\right)<\infty\right\}.$$

Equivalently, $f\in L^p_{\text{weak}}$ if and only if $|\{x:|f(x)|>\lambda\}|\lesssim\lambda^{-p}$. Note that the quantity in (2.1) does not define a norm. This is the reason we append the asterisk to the usual norm notation.

To make a side-by-side comparison with the usual L^p norm, we note that

$$\begin{aligned}\|f\|_{L^p}&=\left(\iint_{0\le\lambda<|f(x)|}p\lambda^{p-1}d\lambda dx\right)^{1/p}\\&=\left(\int_0^\infty|\{|f|>\lambda\}|p\lambda^p\frac{d\lambda}{\lambda}\right)^{1/p}\\&=p^{1/p}\left\|\lambda|\{|f|>\lambda\}|^{1/p}\right\|_{L^p\left((0,\infty),\frac{d\lambda}{\lambda}\right)}\end{aligned}$$

and, with the convention that $p^{1/\infty}=1$,

$$\|f\|^*_{L^p_{\text{weak}}}=p^{1/\infty}\left\|\lambda|\{|f|>\lambda\}|^{1/p}\right\|_{L^\infty\left((0,\infty),\frac{d\lambda}{\lambda}\right)}.$$

This suggests the following definition.

Definition 2.1. For $1\le p<\infty$ and $1\le q\le\infty$ we define the Lorentz space $L^{p,q}\left(\mathbb{R}^d\right)$ as the space of measurable functions f for which

$$\|f\|^*_{L^{p,q}}:=p^{1/q}\left\|\lambda|\{|f|>\lambda\}|^{1/p}\right\|_{L^q\left(\frac{d\lambda}{\lambda}\right)}<\infty.$$

From the discussion above, we see that $L^{p,p} = L^p$ and $L^{p,\infty} = L^p_{\text{weak}}$.

Lemma 2.2. *Given $f \in L^{p,q}$, we write $f = \sum f_m$, where*

$$f_m(x) := f(x)\chi_{\{x: 2^m \le |f(x)| < 2^{m+1}\}}.$$

Then

$$\|f\|^*_{L^{p,q}} \approx_{p,q} \Big\| \|f_m\|_{L^p_x(\mathbb{R}^d)} \Big\|_{\ell^q_m(\mathbb{Z})}$$

In particular, $L^{p,q_1} \subseteq L^{p,q_2}$ whenever $q_1 \le q_2$.

2.2. **Interpolations.**

Definition 2.3 ([11, Chapter 1.3.1])**.** For an interpolation couple (A_0, A_1), we have

$$K(t,a) := \inf_{a = a_0 + a_1} \|a_0\|_{A_0} + t\|a_1\|_{A_1}$$

is an equivalent norm on $A_0 + A_1$ for $0 < t < \infty$, where $t \in (0,\infty)$ is a scaling parameter.

Definition 2.4 ([11, Chapter 1.3.2])**.** For $0 < \theta < 1$, $1 \le q < \infty$, we define

$$(A_0, A_1)_{\theta,q} := \left\{ a : a \in A_0 + A_1, \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta} K(t,a) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

Theorem 2.5 ([11, Chapter 1.18.4])**.** *Let $1 \le p_0, p_1 < \infty$, $0 < \theta < 1$,*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and (A_0, A_1) be an interpolation couple. Then

$$(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,p} = L^p((A_0, A_1)_{\theta,p}).$$

Theorem 2.6 ([11, Chapter 1.18.6])**.** *Let $0 < \theta < 1, 1 < p_0, p_1 < \infty, p_0 \neq p_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$(L^{p_0}(A), L^{p_1}(A))_{\theta,q} = L^{p,q}(A).$$

Remark 2.7. In general, you would expect the interpolation of L^p type spaces gives you Lorentz spaces and the interpolation of Sobolev type spaces gives you Besov spaces.

See [3, Lemma 6.1] for the bilinear interpolation we need.

2.3. **Strichartz estimates and its Lorentz refinement version.** For the linear wave equation

$$\Box v = h, \quad v(t=0) = f, \quad \partial_t v(t=0) = g,$$

we recall the endpoint Strichartz estimates from Keel-Tao [3]

$$\|v\|_{L^2 L^{\frac{2(m-1)}{m-3}}} + \|v\|_{C^0 \dot{H}^\gamma} + \|\partial_t v\|_{C^0 \dot{H}^{\gamma-1}} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}},$$

where $\gamma = \frac{m+1}{2(m-1)}$.

The proof of [7, Section 5] combine this with the interpolation

$$(L^2 L^{\frac{2(m-1)}{m-3}}, L^2 \dot{W}^{\frac{2(m-1)}{m-3}})_{1/(2(m-1)),2} \hookrightarrow L^2_t L^{(2m,2)}_x,$$

where $k=(m^2-4m+1)/2(m-1)$ to obtain

$$\|v\|_{L_t^2L_x^{(2m,2)}}+\|dv\|_{C^0\dot{H}^{m/2-2}}\lesssim\|f\|_{\dot{H}^{m/2-1}}+\|g\|_{\dot{H}^{m/2-2}}. \tag{2.2}$$

Remark 2.8. I'm not quite sure whether their argument is the correct way of understanding this estimate and the interpolation, but it seems like it's a mixed use of Gagliardo-Nirenberg interpolation and the interpolation results in [11].

For the estimates (2.2), it is actually already proved in [3] although they did not explicit state this in their main theorems. This Lorentz space regularity is achieved in the interpolation proof for the endpoint bilinear estimates in [3]. More specifically, they would obtain

$$T:L_t^2L_x^{r',2}\times L_t^2L_x^{r',2}\to(l_1^0)_j,$$

which corresponds to [3, Section 6]. Here T is the bilinear operator

$$\begin{aligned}T(F,G)&=\iint_{s<t}\langle(U(s))^*F(s),(U(t))^*G(t)\rangle\;ds\,dt,\\ T_j(F,G)&=\int_{t-2^{j+1}<s\le t-2^j}\langle(U(s))^*F(s),(U(t))^*G(t)\rangle\;ds\,dt\end{aligned}$$

and

$$(q,r)=\left(2,\frac{2\sigma}{\sigma-1}\right)=\left(2,\frac{2(m-1)}{m-3}\right),\quad\sigma=\frac{m-1}{2}. \tag{2.3}$$

Therefore,

$$\left\|\int(U(s))^*F(s)\,ds\right\|_{\dot{H}^{-\gamma}}\lesssim\|F\|_{L_t^2L_x^{r',2}}$$

and hence

$$\|U(t)f\|_{L_t^2L_x^{r,2}}\lesssim\|f\|_{\dot{H}^\gamma},$$

where γ satisfies

$$\frac{1}{2}+\frac{m}{r}=\frac{m}{2}-\gamma,\frac{1}{2}+\frac{m}{r'}=2+\frac{m}{2}+\gamma.$$

However, though it is overkill to prove the nonendpoint case in this bilinear manner, it is worth doing if we want this Lorentz gain. In particular, we would get the same estimates with (q,r) not being the specific pair in (2.3). By choosing $\gamma=\frac{m}{2}-1$, we obtain

$$\|U(t)f\|_{L_t^2L_x^{2m,2}}\lesssim\|f\|_{\dot{H}^{m/2-1}},$$

where $2m$ can be seen from scaling. This will lead to (2.2). By Duhamel's principle, we obtain

$$\|v\|_{L_t^2L_x^{2m,2}}+\|dv\|_{C^0\dot{H}^{m/2-2}}\lesssim\|f\|_{\dot{H}^{m/2-1}}+\|g\|_{\dot{H}^{m/2-2}}+\|f\|_{L_t^1\dot{H}_x^{m/2-2}}. \tag{2.4}$$

3. Proof of Theorem 1.1

In this section, we use d to denote ∂_α, ∇ to denote ∂_j, D to denote covariant derivative unless otherwise specified.

3.1. **Gauge choices.** By applying the result in [12], we are able to choose Coulomb gauge to transform the equation solved by A_β to be a system of elliptic equations. To present the idea, we first use the assumptions on N to choose a smooth orthonormal frame $\{\bar{e}_a \circ u\}_{1\le a\le k}$ for the pullback bundle u^*TN. Moreover, we may freely rotate this frame at any $z=(t,x)\in\mathbb{R}^{m+1}$ with a matrix $(R_a^b)=(R_a^b(z))\in SO(k)$, thus obtaining the frame

$$e_a = R_a^b(\bar{e}_b\circ u), \quad 1\le a\le k.$$

Expressing $du=\partial^\alpha u\partial_\alpha$ as

$$du = q^a e_a, \quad q = q_\alpha\, dx^\alpha,$$

where $q=(q_\alpha^a)$ is a $k\times(m+1)$ matrix. Thus

$$|du|^2 = |q|^2 = \sum_\alpha |q_\alpha|^2.$$

In particular, all the Lebesgue spaces for du are well-defined and hence by interpolation, Lorentz spaces are also well-defined. Also, they are independent of the choice of "gauges" (R_a^b) and coincide with the L^p norm of du in the extrinsic representation of u as a map $u:\mathbb{R}^{m+1}\to N\subset\mathbb{R}^n$. However, when it comes to higher derivatives, there will be second fundamental forms involved to measure the difference between extrinsic derivatives and the intrinsic ones.

Let D be the pullback covariant derivative, we express the connection 1-form

$$D_\alpha e_a = A^b_{a,\alpha} e_b, \quad 1\le a\le k,$$

where the comma notation does not mean anything related to derivatives but just a separation between different indices. Here $A=A_\alpha dx^\alpha$ is a matrix-valued 1-form. Now we compute

$$\begin{aligned} R(\partial_\alpha u,\partial_\beta u)e_a &= D_\alpha D_\beta e_a - D_\beta D_\alpha e_a = D_\alpha(A^b_{a,\beta}e_b) - D_\beta(A^b_{a,\alpha}e_b) \\ &= (\partial_\alpha A^c_{a,\beta} - \partial_\beta A^c_{a,\alpha} + A^c_{b,\alpha}A^b_{a,\beta} - A^c_{b,\beta}A^b_{a_\alpha})e_c := F^c_{a,\alpha\beta}e_c, \end{aligned}$$

or one may drop the indices to write for short :

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = F_{\alpha\beta} = R(\partial_\alpha u, \partial_\beta u). \tag{3.1}$$

In fact, we want to select a specific bunch of R_a^b so that the corresponding A's are Coulomb gauge, which satisfies

$$\sum_{i=1}^m \partial_i A_i = 0.$$

The existence of Coulomb gauge is proved in [12] under the assumption that $\|F\|_{L^{m/2}}$ is small enough. In our case, given the expression for F and the boundedness on R in the assumption, it suffices to require $\|du\|_{L^m}\lesssim\|du\|_{\dot{H}^{m/2-1}}<\varepsilon_0$ to be small enough. By (3.1), we choose $\alpha=i$ and take ∂_i to obtain $m+1$ elliptic equations :

$$\Delta A_\beta + \partial_i[A_i, A_\beta] = \partial_i F_{i\beta} = \partial_i(R(\partial_i u, \partial_\beta u)), \quad 0\le\beta\le m.$$

In fact, the results in [12] gives you more information about A, which satisfies

$$\|A\|_{L^m} \lesssim \|A\|_{W^{1,m/2}} \lesssim \|F+[A,A]\|_{L^{m/2}} \lesssim \|R\|_{L^\infty}\|du\|^2_{L^m} + \|A\|^2_{L^m},$$

where the second step uses elliptic estimates (boundedness of Riesz transform in $L^{m/2}$) and the constant C is uniform. Then by bootstraping, we know

$$\|A\|_{L^m} \lesssim \|R\|_{L^\infty}\|du\|_{L^m}^2 \lesssim \|du\|_{\dot{H}^{m/2-1}}^2 \leq C\varepsilon_0.$$

The smallness of $\|A\|_{L^m}$ is always assumed, and this estimate is contained in [12, Lemma 2.5]. Therefore, going forward, we always assume $\|du\|_{\dot{H}^{m/2-1}} < \varepsilon_0$ to be small enough and prove an a priori bound under this assumption to close the argument.

By elliptic estimates, chain rule and Sobolev embeddings, we can easily prove the following lemma.

Lemma 3.1. *(Under the assumption that du having sufficiently small L^m norm.) For any time t, we have the following :*

a) For $1 \leq k \leq m/2$, $k \in \mathbb{N}$,

$$\frac{k-1}{\frac{m}{2}-1} \leq a \leq 1, \quad \frac{1}{r} = \frac{k}{m} + \frac{a}{2m},$$

there holds

$$\|\nabla^k A\|_{L^r} + \|\nabla^{k-1}\partial_0 A\|_{L^r} \leq C\|\nabla^{k-1}F\|_{L^r} \leq C\|du\|_{L^{2m}}^{2-a}\|du\|_{\dot{H}^{m/2-1}}^a.$$

b) By Gagliardo-Nirenberg interpolation,

$$\|\nabla^{k-1}A^2\|_{L^r} \leq C\|A\|_{L^m}\|\nabla^k A\|_{L^r}.$$

c) For any $l \leq [m/2], l \in \mathbb{N}$, there holds

$$\sum_{k=0}^{l}\|\nabla^k F\nabla^{l-k}du\|_{L^2} \leq C\|du\|_{L^{2m}}^2\|du\|_{\dot{H}^{l+1}}.$$

d) We have the following estimates regarding the Lorentz spaces :

$$\|A\|_{L^\infty} \leq C\|du\|_{L^{2m,2}}^2.$$

Remark 3.2. Here, ∇ denotes any spatial covariant derivative D_j. In the proof of this lemma, we need to apply Gagliardo-Nirenberg interpolation several times, such as

$$\|\nabla^{k_0}R\|_{L^{m/k_0}} \lesssim \|R\|_{L^\infty}\|\nabla^{m/2}R\|_{L^2} \lesssim \||du|^{m/2}\|_{L^2} \lesssim \|du\|_{L^m}^{2/m} \lesssim 1.$$

3.2. **Equivalence of norms.** By definition of du,

$$du = q^a e_a,$$

we know that $|du| = |q|$. We also claim that for higher derivatives, due to the presence of second fundamental form, the norms are different. Fortunately, under assumptions of boundedness of B and smallness of ε_0, the H^s norms are equivalent.

For any $W \in \mathfrak{X}(u^*TN)$, suppose the coordinates in the frame $\{e_a\}$ are given by

$$W = Q^a e_a = Qe, \quad \|W\|_{L^2} = \|Q\|_{L^2}.$$

The extrinsic partial derivatives of W, $\partial_k W$, can be computed as follows

$$D_k W = \partial_k W + B(u)(\partial_k u, W) = (\partial_k Q + AQ)e,$$

which implies

$$\partial_k W = (\partial_k Q + AQ)e - B(u)(\partial_k u, W).$$

Therefore,

$$|\|\partial W\|_{L^2} - \|\partial Q\|_{L^2}| \lesssim \|AQ\|_{L^2} + \|(du)Q\|_{L^2} \lesssim (\|A\|_{L^m} + \|du\|_{L^m})\|pQ\|_{L^2} \lesssim \|\partial Q\|_{L^2},$$

where we need to assume $\|du\|_{L^m}$ small enough so that the gauge choices are justified.

3.3. **A priori bounds.** In order to derive existence, we need some a priori bounds. Recall

$$D_\alpha e_a = A^b_{a,\alpha} e_b, \quad 1 \le a \le k.$$

We write

$$\partial_\beta u = q^a_\beta e_a, \quad du = q^a e_a$$

so that

$$0 = D_\alpha \partial_\beta u - D_\beta \partial_\alpha u = \partial_\alpha q^a_\beta e_a + q^a_\beta A^b_{a,\alpha} e_b - \partial_\beta q^a_\alpha e_a - q^a_\alpha A^b_{a,\beta} e_b = (\partial_\alpha q^b_\beta + q^a_\beta A^b_{a,\alpha} - \partial_\beta q^b_\alpha - q^a_\alpha A^b_{a,\beta}) e_b.$$

Denote

$$D_\alpha q_\beta = (\partial_\alpha + A_\alpha) q_\beta, \tag{3.2}$$

then we have

$$0 = (D_\alpha q_\beta - D_\beta q_\alpha) e,$$

that is, we have $D_\alpha q_\beta - D_\beta q_\alpha = 0$.

Moreover, by noting that wave map equation (1.1) is equivalent to $D^\alpha q_\alpha = 0$ and hence

$$0 = D_\beta D^\alpha q_\alpha = D^\alpha D_\beta q_\alpha + F^\alpha_\beta q_\alpha = D^\alpha D_\alpha q_\beta + F^\alpha_\beta q_\alpha.$$

We can further expand this by using (3.2) to obtain

$$(\partial_t^2 - \Delta) q_\beta = 2A^\alpha \partial_\alpha q_\beta + (\partial^\alpha A_\alpha) q_\beta + A^\alpha A_\alpha q_\beta + F^\alpha_\beta q_\alpha,$$

where we denote the right hand side by h_β.

We can estimate q in terms of the initial data and h by applying the Strichartz estimates in the Lorentz spaces setting (2.4) on some time interval $[0, T]$ so that $\|du\|_{\dot H^{m/2-1}}$ is small enough, uniformly for $0 < t < T$. Thanks to the equivalence of norms on such time interval, we obtain

$$\|du\|_{C^0_t \dot H^{m/2-1}} + \|du\|_{L^2_t L^{2m,2}_x} \lesssim \|du(0)\|_{\dot H^{m/2-1}} + \|h\|_{L^1_t \dot H^{m/2-2}_x} \lesssim \|u_0\|_{\dot H^{m/2}} + \|u_1\|_{\dot H^{m/2-1}} + \|h\|_{L^1_t \dot H^{m/2-2}_x}. \tag{3.3}$$

Remark 3.3. Heuristically, the difficulty of how to estimate the source term in $L^1 L^2$ appears again (we just put the derivatives $|D|$ implicitly in the norm). The resolution of this in [7] is to use the Lorentz spaces, where the key estimate is Lemma 3.1 (d) since when one encounter $\|A\|_{L^\infty} \|\partial q\|_{\dot H^{m/2-2}}$ with no derivatives falling on A, the term $\|A\|\|_{L^1 L^\infty}$ is impossible to be bounded without Lorentz spaces introduced, as we can see below.

Now we estimate the term $\|h\|_{L^1_t\dot{H}^{m/2-2}_x}$. First, we consider the case when m is even. We compute

$$\begin{aligned}\|h\|_{\dot{H}^{m/2-2}} &\le 2\|A\partial q\|_{\dot{H}^{m/2-2}} + \|(\partial A)q\|_{\dot{H}^{m/2-2}} + \|A^2q\|_{\dot{H}^{m/2-2}} + \|Fq\|_{\dot{H}^{m/2-2}}\\ &\lesssim \|A\|_{L^\infty}\|q\|_{\dot{H}^{m/2-1}} + \sum_{k_1+k_2=m/2-2}\|\nabla^{k_1}q\|_{L^{r_1}}\left(\|\nabla^{k_2+1}A\|_{L^{r_2}} + \|\nabla^{k_2}A^2\|_{L^{r_2}} + \|\nabla^{k_2}F\|_{L^{r_2}}\right)\\ &\lesssim \|A\|_{L^\infty}\|q\|_{\dot{H}^{m/2-1}} + \sum_{k_1+k_2=m/2-2}\|\nabla^{k_1}q\|_{L^{r_1}}\|\nabla^{k_2}F\|_{L^{r_2}}\end{aligned}$$

where $\frac{1}{2}=\frac{1}{r_1}+\frac{1}{r_2}$ in the second inequality, we apply Lemma 3.1 (a), (b) and the smallness of $\|A\|_{L^m}$ in the third step.

Moreover, from Lemma 3.1 (d), we obtain

$$\|h\|_{\dot{H}^{m/2-1}} \lesssim \|du\|^2_{L^{2m,2}}\|q\|_{\dot{H}^{m/2-1}} + \|q\|^{1-a_1}_{L^{2m}}\|q\|^{a_1}_{\dot{H}^{m/2-1}}\|du\|^{1-a_2}_{L^{2m}}\|du\|^{a_2}_{\dot{H}^{m/2-1}},$$

where

$$\frac{1}{r_1}=\frac{k_1}{m}+a_1(\frac{1}{2}-\frac{m/2-1}{m})+\frac{1-a_1}{2m}=\frac{k_1}{m}+\frac{1+a_1}{2m},\quad \frac{1}{r_2}=\frac{k_2+1}{m}+\frac{a_2}{2m},\quad k_1+k_2=\frac{m}{2}-2,\quad \frac{1}{2}=\frac{1}{r_1}+\frac{1}{r_2}.$$

From these, we notice that $a_1+a_2=1$ so we obtain from the equivalence of norms between du and q that

$$\|h\|_{\dot{H}^{m/2-1}} \lesssim \|du\|^2_{L^{2m,2}}\|du\|_{\dot{H}^{m/2-1}}.$$

For m odd, we need to worry about the top order half derivative. We can achieve the desired bound by interpolation.

Therefore, by combining this with (3.3), we obtain

$$\|du\|_{C^0_t\dot{H}^{m/2-1}} + \|du\|_{L^2_tL^{2m,2}_x} \le C\left(\|u_0\|_{\dot{H}^{m/2}} + \|u_1\|_{\dot{H}^{m/2-1}} + \|du\|^2_{L^2L^{2m,2}}\|du\|_{C^0_t\dot{H}^{m/2-1}}\right).$$

and hence

$$\|du\|_{C^0_t\dot{H}^{m/2-1}} + \|du\|_{L^2_tL^{2m,2}_x} \le \tilde{C}\varepsilon_0$$

when $\|u_0\|_{\dot{H}^{m/2}} + \|u_1\|_{\dot{H}^{m/2-1}} \le \varepsilon_0$ sufficiently small. This can be proved by a bootstrap argument by assuming

$$\|du\|_{L^2L^{2m,2}} + \|du\|_{C^0_t\dot{H}^{m/2-1}} \le \varepsilon$$

and to show

$$\|du\|_{L^2L^{2m,2}} + \|du\|_{C^0_t\dot{H}^{m/2-1}} \le \frac{1}{2}\varepsilon,$$

where $\varepsilon, \varepsilon_0$ both small enough. This is exactly the same as the classical example for bootstrap in [9]. Therefore, we obtain the desired global a priori bound

$$\|du\|_{C^0_t\dot{H}^{m/2-1}} + \|du\|_{L^2_tL^{2m,2}_x} \le C(\|u_0\|_{\dot{H}^{m/2}} + \|u_1\|_{\dot{H}^{m/2-1}})$$

when $\|u_0\|_{\dot{H}^{m/2}} + \|u_1\|_{\dot{H}^{m/2-1}} \le \varepsilon_0$ small enough. Here, global means that C is uniformly in $t\in\mathbb{R}$ and the bound holds on any time interval the solution exists.

3.4. **Higher regularity results.** To prove the global existence, we need to prove a higher regularity result. First,

$$\frac{d}{dt}\frac{1}{2}\|du(t)\|_{L^2}^2 = \int \langle D_0 D^\alpha u, D_\alpha u\rangle = \int \langle D_0 u, D^\alpha D_\alpha u\rangle = 0,$$

so $u \in H^1$.

Now we show that smooth solutions u of (1.2) with

$$\|du\|_{C_t^0 \dot{H}^{m/2-1}} + \|du\|_{L_t^2 L_x^{2m,2}}$$

sufficiently small remain bounded in terms of the data $(u_0, u_1) \in H^l \times H^{l-1}(\mathbb{R}^m; TN)$ for any $l \geq m/2$. This can be easily seen from the standard energy estimates.

Let $v = D_j u = \nabla u$, where ∇ denotes the covariant derivatives in spatial directions. From

$$0 = D_j D_\alpha \partial^\alpha u = D_\alpha D_j \partial^\alpha u + R(\partial_j u, \partial_\alpha u)\partial^\alpha u = D_\alpha D^\alpha v + R(\partial_j u, \partial_\alpha u)\partial^\alpha u,$$

we obtain

$$D^\alpha D_\alpha v = F du, \quad F_{\alpha\beta} = R(\partial_\alpha u, \partial_\beta u). \tag{3.4}$$

Upon multiplying this by $D_0 v$ and integrating over $\mathbb{R}^m$, we find

$$\begin{aligned}\frac{d}{dt}\|Dv\|_{L^2}^2 &\lesssim \int |\langle D_0 D_\alpha v, D^\alpha v\rangle| \lesssim \int |\langle D_0 v, D_\alpha D^\alpha v\rangle| + |\langle R(\partial_0 u, \partial_\alpha u)v, D^\alpha v\rangle| \\ &\lesssim \|F du Dv\|_{L^1} \lesssim \|F\|_{L^m}\|du\|_{L^{\frac{2m}{m-2}}}\|Dv\|_{L^2} \lesssim \|F\|_{L^m}\|Dv\|_{L^2}^2 \lesssim \|du\|_{L^{2m}}^2\|Dv\|_{L^2}^2,\end{aligned}$$

where we use the embedding $\dot{H}^1 \subset L^{\frac{2m}{m-2}}$. By integrating this and the equivalence of covariant and extrinsic L^p, we have the boundedness of $\|u\|_{H^2}$.

We differentiate (3.4) once more to compute

$$\nabla(F du) = \nabla D^\alpha D_\alpha v = D^\alpha \nabla D_\alpha v + F D_\alpha v = D^\alpha D_\alpha \nabla v + F Dv,$$

which implies

$$\frac{d}{dt}\|D\nabla v\|_{L^2}^2 \leq \int |\langle D_0 \nabla v, \nabla(F du) + F Dv\rangle| \lesssim \|D\nabla v\|_{L^2}\|\nabla(F du)\|_{L^2}.$$

In general, for higher derivatives, we would derive

$$\frac{d}{dt}\|\nabla^{l-1} D\nabla v\|_{L^2}^2 \lesssim \sum_{k=0}^{l} \|\nabla^k F \nabla^{l-k} du\|_{L^2}\|\nabla^{l-1} D\nabla v\|_{L^2}. \tag{3.5}$$

Moreover, for $l = 1, \cdots, [m/2]$, we can estimate by Lemma 3.1 to find

$$\sum_{k=0}^{l} \|\nabla^k F \nabla^{l-k} du\|_{L^2} \lesssim \|du\|_{L^{2m}}^2 \|du\|_{\dot{H}^{l+1}} \lesssim \|du\|_{L^{2m}}^2 \|\nabla^{l-1} D\nabla v\|_{L^2},$$

which gives the boundedness of $\|du\|_{\dot{H}^{l+1}}$ for $l \leq [m/2]$. (This is already enough to conclude the global existence result in $H^{m/2}$.) In particular, $du \in L_t^\infty H_x^{[m/2]+1} \hookrightarrow L^\infty$.

For larger l, although we cannot directly apply the estimates in Lemma 3.1, it is much simpler due to the boundedness of du. (This part is just to prove the last line in the statement of Theorem 1.1.) For $l = [m/2] + 1$, the only difference happens when $k = 0$ or $k = l$, where the first case is direct bounded by $\|F\|_{L^\infty}\|du\|_{\dot{H}^{l+1}}$ and is easy to bound due to $du \in L^\infty$.

For the second case, one needs to estimate $\|\nabla^l F du\|_{L^2}$ by $\|du\|_{\dot{H}^{l+1}}$. Recall that ∇ denotes any spatial covariant derivative, so

$$|(\nabla^{[m/2]+1}F)| \lesssim ||du|^{[m/2]+3}|.$$

Since $du \in H^{[m/2]+1} \subset L^\infty \cap L^2$, we can estimate

$$\|\nabla^l F du\|_{L^2} \lesssim \||du|^{[m/2]+4}\|_{L^2},$$

One can find an appropriate $2 \leq p \leq \infty$ and $\theta < 1$ such that

$$\|du\|_{L^p} \lesssim \|D^{[m/2]+2}du\|_{L^2}^{\theta}\|du\|_{L^2}^{1-\theta},$$

where $\frac{1}{2} - \frac{1}{p} = \frac{m+4}{2m}\theta$. We choose $\theta = \frac{4}{m+4}$, then $1/p = (m-4)/(2m)$, then if one needs $\|D^{[m/2]+2}du\|_{L^2}^2$ in the right hand side of the estimates, one would expect

$$\frac{1}{2} > \frac{k}{p}, \quad 2 \leq p \leq \infty$$

with $k = \frac{m+4}{2}$ which is possible to achieve. For $l = [m/2]+2$, one also needs to consider $k = 1$ and $k = l-1$, but this time we know $\nabla du \in L^\infty$, so we can achieve higher regularity (not necessarily uniform in time) by bootstraping.

Finally, we use the equation (1.2) to obtain the higher time derivatives and mixed space-time derivatives.

3.5. **Global existence.** First, we approximate the initial data $(u_0, u_1) \in H^{m/2} \times H^{m/2-1}(\mathbb{R}^m; TN)$ by smooth initial data $(u_0^{(k)}, u_1^{(k)})$ with some compact support property indicated in (1.3), which ensures local existence. In other words, $(u_0^{(k)}, u_1^{(k)}) \to (u_0, u_1)$ in $H^{m/2} \times H^{m/2-1}(\mathbb{R}^m; TN)$ as $k \to \infty$. Suppose the initial energy

$$\|u_0\|_{\dot{H}^{m/2}}^2 + \|u_1\|_{\dot{H}^{m/2-1}}^2 \leq \varepsilon_0$$

is sufficiently small, then the local solutions $u^{(k)}$ satisfies our a priori bounds (uniformly on the time of existence)

$$\|du^{(k)}\|_{C_t^0 \dot{H}^{m/2-1}} + \|du^{(k)}\|_{L_t^2 L^{2m_x}} \lesssim \|u_0^{(k)}\|_{\dot{H}^{m/2}} + \|u_1^{(k)}\|_{\dot{H}^{m/2-1}} \leq C\varepsilon_0$$

and also the higher regularity results. Therefore, if we choose ε_0 so that $\varepsilon_0, C\varepsilon_0$ are sufficiently small, then we know that $u^{(k)}$ can be extended to a solution for all time. (Suppose by contradiction that the solution only exists on $[0, T]$, then one can construct solution by local existence in $H^{m/2+1}$ at T since $\|du^{(k)}\|_{H^{[m/2]+1}}$ is finite.)

Hence, by higher regularity results, we know $\|du^{(k)}\|_{H_{loc}^{m/2}(\mathbb{R}^{m+1})}$ is bounded uniformly (the subscript loc is added in since we consider both space and time) and hence $u^{(k)} \to u$ weakly in $H_{loc}^{m/2}(\mathbb{R}^{m+1})$ by passing to a subsequence. Moreover,

$$\|du\|_{C_t^0 \dot{H}^{m/2-1}} + \|du\|_{L_t^2 L_x^{2m}} \lesssim \|u_0\|_{\dot{H}^{m/2}} + \|u_1\|_{\dot{H}^{m/2-1}}.$$

By Rellich compactness theorem $m/2 - 1 > 1$, $du^{(k)} \to du$ converges in L^2 and hence pointwisely almost everywhere, which allows us to pass to the limit in the equation (1.1) so that u solves it with the desired initial data.

3.6. **Uniqueness.** In order to prove uniqueness, we use the extrinsic form (1.2). Suppose u, v are two solutions in $H^{m/2}$ with the same initial data, and also suppose $\|du\|_{L^2L^{2m}} + \|dv\|_{L^2L^{2m}}$ is finite. Then $w = u - v$ satisfies

$$\Box w = (B(u) - B(v))(\partial_\alpha u, \partial^\alpha u) + B(v)(\partial_\alpha u + \partial_\alpha v, \partial^\alpha w).$$

Multiplying this by w_t and do the usual energy estimates, one is able to show

$$\frac{d}{dt}\|dw\|_{L^2}^2 \lesssim \left(\|du\|_{L^{2m}}^2 + \|dv\|_{L^{2m}}^2\right)\|dw\|_{L^2}^2,$$

where we use the geometric structure to exploit the orthogonality condition

$$\langle B(u)(\cdot,\cdot), u_t\rangle = 0 = \langle B(v)(\cdot,\cdot), v_t\rangle.$$

Therefore, the solution is unique since $dw(0) = 0$.

4. Related works on wave maps

4.1. **An overview of Tao's method.** We follow [10], [8]. Basically, [8] still wants to put the nonlinearity in L^1L^2 and apply Strichartz estimates. To get around the failure discussed in Section 1.2, we use Littlewood-Paley theory.

Since nonlinear estimates are difficult to prove, we seek to reduce to linear estimates by using paradifferential calculus. The basic principle is to transform the equation into an infinite system

$$P^{lin}(\phi_{<k})\phi_k = \text{ error},$$

where $\phi_{<k} = P_{<k-10}\phi$, $\phi_k = P_k\phi$. For wave maps with sphere target, we apply P_k to (1.5)

$$\Box(P_k\phi^i) = P_k(\phi\partial^\mu\phi\partial_\mu\phi),$$

where we still need to split the expression inside the projection P_k on the right hand side. We would expect terms like

$$P_k(\phi_{k_1}\partial^\mu\phi_{k_2}\partial_\mu\phi_{k_3}),$$

where the only problematic term is when $k_1 < \min\{k_2, k_3\}$, that is, the undifferentiated term is the smallest in frequency. Heuristically, we usually expect $(\nabla\phi)\psi$ is very small compared to $\phi\nabla\psi$, where ψ is much rougher, i.e. higher frequency due to the high oscillation.

It turns out that we can linearize it into

$$\Box\phi_k^i = 2\phi_{<k}^i\partial^\alpha\phi_{<k}^j\partial_\alpha\phi_k^j + \text{ error},$$

where the error is in the sense of L^1L^2, see [8, Section 4] for the details. However, this is not satisfactory enough. One uses the geometric structure again $\phi^j\partial_\alpha\phi^j = 0$, so we would expect good control for $\phi_{<k}^j\partial_\alpha\phi_k^j$, which allows us to write the above equation as

$$\Box\phi_k^i = 2(\phi_{<k}^i\partial^\alpha\phi_{<k}^j - \phi_{<k}^j\partial^\alpha\phi_{<k}^i)\partial_\alpha\phi_k^j + \text{ error}.$$

In this way, we obtain anti-symmetric matrices

$$(A_{<k}^\alpha)^{ij} = \left(\phi_{<k}^i\partial^\alpha\phi_{<k}^j - \phi_{<k}^j\partial^\alpha\phi_{<k}^i\right).$$

See [8, Section 5] for details.

The importance of the anti-symmetry can be seen from the following ODE analogue :

$$\ddot{\psi} = 2A_0\dot{\psi}. \tag{4.1}$$

Let $U(t)$ be a matrix-valued function solving the ODE

$$\dot{U}(t) = A_0 U(t),$$

with A_0 anti-symmetric matrix and $U(0) = I$. Thanks the anti-symmetry,

$$\frac{d}{dt} U_{(j)}(t) \cdot U^*_{(i)}(t) = 0,$$

where $U_{(j)}$ denotes the j-th column of U, that is, U remains orthogonal for all time. Suppose we make a linear change of variable

$$\psi = U(t) w,$$

where the orthogonality allows us to reduce the estimate for ψ to w. By plugging this into (4.1) and ignore zeroth order term for w, which can be treated as an error term, we obtain

$$\ddot{w} = 0.$$

Therefore, we seek for a gauge transformation

$$\phi_k \mapsto U_{<k} \phi_k$$

so that $U_{<k}$ is orthogonal and satisfies

$$\partial^\alpha U_{<k} = A^\alpha_{<k} U_{<k}.$$

Unfortunately, this is unable to achieve unless some further constraints hold for A. Instead, we look for approximate solutions by defining them inductively by

$$U_k := (\phi^i_k \phi^j_{<k} - \phi^j_{<k} \phi^i_k) U_{<k}, \quad U_{<k} = I + \sum_{-M<k'<k} U_{k'},$$

where U_k then satisfies

$$\partial_\alpha U_k \approx A_\alpha U_{<k}.$$

One would expect almost orthogonality condition for

$$U = I + \sum_{-M<k} U_k,$$

and finally obtain

$$\Box(U_{<k}\phi_k) = \text{ error},$$

which allows us to use linear estimates for $U_{<k}\phi_k$ and closed the loop.

4.2. **Null structure.** [5] exploits the null structure in the nonlinearity to show the global existence with small, compactly supported smooth data by applying the vector field method. This is discussed in [10]. In [6], the authors take advantage of the null structure in a different way, which works better for translation invariant space. Also, compared to the preceding ones, we do not need the pointwise bound and so we don't need to assume decay (compact support) in this setup. This idea can be summarized as follows. For $\Box\phi = Q_0(\phi, \phi)$ with nonlinearity denoted by Q_0, we make a Fourier transform

$$(\tau^2 - |\xi|^2)\widetilde{\phi} = \int_{\tau=\tau_1+\tau_2, \xi=\xi_1+\xi_2} m(\tau_1, \xi_1, \tau_2, \xi_2) \widetilde{\phi}(\tau_1, \xi_1) \widetilde{\phi}(\tau_2, \xi_2)\, d\tau_1\, d\xi_1,$$

where in the integrand, things got worst when along the light cone, that is, $|\tau_1| = |\xi_1|$, $|\tau_2| = |\xi_2|$. In view of the left hand side, the worst case is to have $|\tau| = |\xi|$. When all these happen at the same time, we say that is has resonant interactions. However, when we have the null structure, the m we have will give some cancellations in the worst case scenario so that we actually don't need to worry those anymore. First, from basic geometry, resonant interactions can only happen when (τ_1, ξ_1), (τ_2, ξ_2) are along the same direction on the lightcone.

For example, consider a typical type of nonlinearity with null structure :

$$Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_j \phi \partial_i \psi,$$

which corresponds to $(\xi_1)_i(\xi_2)_j - (\xi_1)_j(\xi_2)_i$ in Fourier space with

$$|(\xi_1)_i(\xi_2)_j - (\xi_1)_j(\xi_2)_i| = |\xi_1||\xi_2|| \sin \angle(\xi_1, \xi_2)|$$

where when the resonant interactions happen, angle is zero so that we would expect cancellations. Also, consider another typical type of nonlinearity with null structure :

$$Q_0(\phi, \psi) = \partial_\mu \phi \partial^\mu \psi = \frac{1}{2}(\Box(\phi\psi) - (\Box\phi)\psi - \phi(\Box\psi)).$$

As we can see from the discussion about $X^{s,b}$ spaces, $\Box$ is invertible away from the light cone, so the worst things happen there. However, this has null structure $m = |\xi_1||\xi_2|O(|\angle(\xi_1, \xi_2)|^2)$ while we assume $|\tau_1| \lesssim |\xi_1|$ and $|\tau_2| \lesssim |\xi_2|$.

References

1. Garth Baker and Alexandre Freire (eds.), *Nonlinear partial differential equations in geometry and physics*, Progress in Nonlinear Differential Equations and their Applications, vol. 29, Birkhäuser Verlag, Basel, 1997, The 1995 Barrett Lectures. MR 1437149
2. Demetrios Christodoulou and A. Shadi Tahvildar-Zadeh, *On the regularity of spherically symmetric wave maps*, Comm. Pure Appl. Math. **46** (1993), no. 7, 1041–1091. MR 1223662
3. Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980. MR 1646048
4. Rowan Killip, *247a notes on lorentz spaces*, available online at : https://www.math.ucla.edu/~killip/247a/Lorentz_space_notes.pdf.
5. S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326. MR 837683
6. Sergiu Klainerman and Matei Machedon, *Estimates for null forms and the spaces $H_{s,\delta}$*, Internat. Math. Res. Notices (1996), no. 17, 853–865. MR 1420552
7. Jalal Shatah and Michael Struwe, *The Cauchy problem for wave maps*, Int. Math. Res. Not. (2002), no. 11, 555–571. MR 1890048
8. Terence Tao, *Global regularity of wave maps. I. Small critical Sobolev norm in high dimension*, Internat. Math. Res. Notices (2001), no. 6, 299–328. MR 1820329
9. ______, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, Local and global analysis. MR 2233925
10. Daniel Tataru, *The wave maps equation*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 2, 185–204. MR 2043751
11. Hans Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam-New York, 1978. MR 503903

12. Karen K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982), no. 1, 31–42. MR 648356

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