SMALL DATA GLOBAL EXISTENCE OF NONLINEAR KLEIN-GORDON EQUATIONS

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CONTENTS

From the work of Klainerman [Kla87] and Shatah [Sha85], it is known that for nonlinear perturbations of the Klein-Gordon equation in $\mathbb{R}^{1+n}$,

$$\Box u - u = F(u, u', u''), \tag{0.1}$$ {eqn:KG_w_F

where $\Box = -\partial_0^2 + \partial_1^2 + \cdots + \partial_n^2$, F vanishes of second order at 0 and F is linear in u'' (this assumption guarantees proper local existence theory), the Cauchy problem with small data in C_0^∞ has a global solution if $n \geqslant 3$. We follow [Hör97, Section 7.3, 7.6 – 7.8].

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1. PRELIMINARIES

We shall prove a crude existence theorem as a motivation for the reason why we need the normal form transformation. {lemma_eng_

Lemma 1.1. *Let u be a solution of the perturbed Klein-Gordon equation*

$$\Box u - u + \gamma^{\mu\nu}(x)\partial_\mu \partial_\nu u = f. \tag{1.1}$$ {eqn:KG_per

If u vanishes for large $|x|$ and $\sum |\gamma^{\mu\nu}| \leqslant \frac{1}{2}$, then

$$\sum_{|\alpha|\leqslant 1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \leqslant 2 \left(\sum_{|\alpha|\leqslant 1} \|\partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|f(s,\cdot)\|_{L^2}\, ds \right) \exp\left(\int_0^t 2\Gamma(s)\, ds \right),$$

Date: October 01, 2024.

where $\Gamma(s) = \sum_{\mu,\nu,\lambda} \sup |\partial_\mu \gamma^{\nu\lambda}(s,\cdot)|$ *and* ∂ *denotes any order* 1 *derivative in* t, x.

_fund_soln}

Lemma 1.2. *If* $u \in C^\infty([0,T]\times\mathbb{R}^n)$ *is rapidly decreasing as* $x \to \infty$ *and is a solution of the Cauchy problem*

$$\Box u - u = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1,$$

then for $0 \leqslant t \leqslant T$, *we have*

$$\sup_x |u(t,x)| \lesssim (1+t)^{-\frac{n}{2}} \sum_{j=0}^{1} \sum_{|\alpha|+j\leqslant n+1} \int |\partial_x^\alpha u_j(x)|\,dx + \sum_{|\alpha|\leqslant n} \iint_{0<s<t} (1+t-s)^{-\frac{n}{2}} |\partial_x^\alpha f(s,x)|\,ds\,dx.$$

Proof. This decay is directly from estimating fundamental solutions and the representation formula

$$u(t,x) = u_0 *_x \partial_t E + u_1 *_x E + f *_{t,x} E.$$

$\square$

_3rd_SDGWP}

Theorem 1.3. *Assume that the function* F *in* (0.2) *is in* C^∞ *and vanishes of third order at* 0, *and the dimension* $n \geqslant 3$. *Then* (0.2) *has a global solution* $C^\infty([0,\infty)\times\mathbb{R}^n)$ *with Cauchy data*

$$u(0) = \varepsilon u_0, \quad \partial_t u(0) = \varepsilon u_1, \quad u_0, u_1 \in \mathscr{S}(\mathbb{R}^n)$$

for ε *sufficiently small.*

Proof. By the local existence theorem (for instance, [Hör97, Theorem 6.4.11]) it suffices to bootstrap the following estimates :

qn:BA-1_KG}

$$\sum_{|\alpha|\leqslant s+1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \leqslant 2M\varepsilon, \quad 0 \leqslant t \leqslant T, \tag{BA-1-KG}$$

qn:BA-2_KG}

$$\sum_{|\alpha|\leqslant \frac{s}{2}+2} \sup_x |\partial^\alpha u(t,x)| \leqslant 2M\varepsilon(1+t)^{-\frac{n}{2}}, \quad 0 \leqslant t \leqslant T, \tag{BA-2-KG}$$

where ∂^α denotes $|\alpha|$-fold derivatives in t or x. Here s is a large integer and will be taken to be $s \geqslant 2n+6$.

Step 1 : Improving (BA-1-KG). Commuting (0.2) with ∂^α, we obtain

nmute_w_dt}

$$(\Box + \gamma^{\mu\nu}\partial_\mu\partial_\nu - 1)\,\partial^\alpha u = f_\alpha, \tag{1.2}$$

where f_α only depends on $\partial^\beta u$ with $|\beta| \leqslant |\alpha|+1$. Here, $\gamma^{\mu\nu}(u,u') := -\frac{\partial F}{\partial u''_{\mu\nu}}$ vanishes of second order at 0. Then it follows from (BA-2-KG) that

n_bound_KG}

$$|\gamma^{\mu\nu}(t,x)| \lesssim (1+t)^{-n}, \quad |\partial\gamma^{\mu\nu}(t,x)| \lesssim (1+t)^{-n}, \tag{1.3}$$

where we remark that the integrability of this rate only requires first order of vanishing of $\gamma^{\mu\nu}$. (Indeed, first order vanishing will lead to $(1+t)^{-\frac{n}{2}}$ rate.) The uniform bound of γ by $\frac{1}{2}$ will follow by taking ε small enough. Hence Lemma 1.1 applies to (1.2), we obtain

$$\sum_{|\beta|\leqslant s+1} \|\partial^\beta u(t,\cdot)\|_{L^2} \lesssim \varepsilon + \sum_{|\beta|\leqslant s} \int_0^t \|f_\beta(s,\cdot)\|_{L^2}\,ds.$$

Expanding f_α and then for each term, there is at most $s+1$ derivatives in total. Therefore, for the (at least) three entries, at most one of them is of order $> \frac{s}{2}+2$ and will be bounded by (BA-1-KG) and the others (at least two) are simply by L^∞ norm (BA-2-KG). Therefore,

lpha_in_KG}

$$\sum_{|\beta|\leqslant s}\int_0^t \|f_\alpha(s,\cdot)\|_{L^2}\,ds \lesssim (M\varepsilon)^3\int_0^\infty (1+t)^{-n}\,dt \lesssim \varepsilon. \tag{1.4}$$

This improves the bootstrap assumption (BA-1-KG).

Step 2 : Improving (BA-2-KG). To prove (BA-2-KG), we rewrite (1.2) into the form

$$(\Box - 1)\partial^\alpha u = F_\alpha, \tag{1.5}$$

where F_α is of third order and contains derivatives of order $\leqslant |\alpha|+2$. We want $|\alpha|+2+n \leqslant s+1$ when $|\alpha| \leqslant \frac{s}{2}+2$, so we take $s \geqslant 2n+6$. Then

$$\int |\partial_x^\beta F_\alpha(t,x)|\,dx \lesssim (M\varepsilon)^3(1+t)^{-\frac{n}{2}} \lesssim \varepsilon(1+t)^{-\frac{n}{2}}, \quad |\beta| \leqslant n, \tag{1.6}$$

{eqn:step_2

since each term in $\partial_x^\beta F_\alpha$ is a product with two factors for which (BA-1-KG) is applicable and one for which (BA-2-KG) is valid. An improvement of (BA-2-KG) then follows from the decay estimates in Lemma 1.2 obtained via fundamental solution thanks to the following integral estimates

$$\int_0^t (1+t-s)^{-\frac{n}{2}}(1+s)^{-\frac{n}{2}}\,ds = 2\int_0^{\frac{t}{2}} (1+t-s)^{-\frac{n}{2}}(1+s)^{-\frac{n}{2}}\,ds \lesssim (1+t)^{-\frac{n}{2}}, \quad n \geqslant 3.$$

□

{rmk_second

Remark 1.4. Examining the proof in Step 1, one actually only requires that F vanishes at 0 of at least second order so that the main estimates (1.3) and (1.4) hold. In particular, (1.4) becomes

$$\sum_{|\beta|\leqslant s}\int_0^t \|f_\alpha(s,\cdot)\|_{L^2}\,ds \lesssim (M\varepsilon)^2\int_0^\infty (1+t)^{-\frac{n}{2}}\,dt \lesssim \varepsilon$$

still holds as long as $n \geqslant 3$.

Remark 1.5. Note that (BA-1-KG) is about the top order derivative estimates and hence we need to justify the number of derivatives we choose to bound in (BA-2-KG) (i.e. $\frac{s}{2}+2$) is not too big. On the other hand, in (BA-2-KG), we can choose s large enough (compared to n) to make it improved. Due to this consideration, when improving (BA-2-KG) using normal form method, we are not afraid of losing derivatives.

Before we introduce normal form transformation, we need to recall theory of bilinear Fourier multipliers.

Definition 1.6. Given a function $m(\xi,\eta) \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, we associate a bilinear operator

$$B_m(u,v)(x) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi-\eta,\eta)\widehat{u}(\xi-\eta)\widehat{v}(\eta)e^{ix\cdot\xi}\,d\eta\,d\xi. \tag{1.7}$$

{eqn:biline

Let $\check{m}$ be the inverse Fourier transform of m in $\mathbb{R}^{2n}$, then B_m can be rewritten as

$$B_m(u,v)(x) = \iint \check{m}(x-y,x-z)u(y)v(z)\,dy\,dz.$$

Example 1.7. The operator can be explicitly written out when m is a polynomial. Suppose $m(\xi,\eta)=\xi^\alpha\eta^\beta$, we compute

$$\begin{aligned}B_m(u,v)(x) &= \frac{1}{(2\pi)^{2n}}\iint(\xi-\eta)^\alpha\eta^\beta\widehat{u}(\xi-\eta)\widehat{v}(\eta)e^{ix\cdot\xi}\,d\eta\,d\xi\\ &= \frac{1}{(2\pi)^{2n}}\iint\widehat{D^\alpha u}(\xi-\eta)\widehat{D^\beta v}(\eta)e^{ix\cdot(\xi-\eta)}e^{ix\cdot\eta}\,d\eta\,d\xi = D_x^\alpha u(x)D_x^\beta v(x),\end{aligned}$$

where $D_x=\frac{1}{i}\partial_x$.

_estimates}

Lemma 1.8. *If* $\check{m}\in L^1(\mathbb{R}^{2n})$, *then*

$$\|B_m(u,v)(\cdot)\|_{L^r}\leqslant\|\check{m}\|_{L^1}\|u\|_{L^p}\|v\|_{L^q},\quad\forall u,v\in\mathscr{S},\quad\frac{1}{r}=\frac{1}{p}+\frac{1}{q},\quad 1\leqslant p,q,r\leqslant\infty.$$

Proof. It is easy to prove the cases $(\frac{1}{p},\frac{1}{q},\frac{1}{r})=(1,0,1),(0,1,1),(0,0,0)$. The general case follows from Riesz-Thorin interpolation. □

nv_F_in_L1}

Lemma 1.9. *Given a symbol* m *such that* $\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)\in L^2(\mathbb{R}^{2n})$ *for any* $|\alpha|,|\beta|\leqslant\frac{n+2}{2}$, *then* $\check{m}\in L^1(\mathbb{R}^{2n})$.

Proof. This follows from Cauchy inequality and L^2-isometry property of Fourier transforms. □

2. Normal form transformation

Now we are ready to introduce the normal form transformation. Shatah [Sha85] proved that Theorem 1.3 is also valid when F just vanishes of second order at 0. The essential idea is to split u into a sum of a term which is quite explicitly determined by u and a term which satisfies a Klein-Gordon equation with right hand side of third order.

Without loss of generality, we could assume that F is independent of ∂_t^2u. Indeed, thanks to the bootstrap assumptions and second order vanishing of nonlinearity, we can solve ∂_t^2 in terms of other derivatives of order $\leqslant 2$ and by plugging this back, we obtain an equation of the same form where ∂_t^2u does not occur in the nonlinearity. Then it follows from Taylor's formula that

_3rd_order}

$$F(u,u',u'')=\sum_{j,k=0}^{1}\mathcal{A}_{jk}(\partial_t^ju,\partial_t^ku)+R(u,u',u''),\tag{2.1}$$

where R vanishes of third order at 0 and $\mathcal{A}_{jk}(\xi,\eta)$ is a real polynomial of degree $\leqslant 2-j$, (resp. $\leqslant 2-k$) in ξ (resp. η). Here, we abuse notations to denote the symbol of the bilinear operator $\mathcal{A}_{jk}$ by $\mathcal{A}_{jk}$.

We want to decompose $u=U+V$ so that U is of the following form

J_decomp_u}

$$U=\sum_{j,k=0}^{1}\mathcal{B}_{jk}(\partial_t^ju,\partial_t^ku),\tag{2.2}$$

where $\mathcal{B}_{jk}$ is a bilinear operator with symbol denoted by the same notation as well. Then we verify that

with_V_LHS}

$$(\Box-1)V=F(u,u',u'')-(\Box-1)U\tag{2.3}$$

is of third order in u if one selects $\mathcal{B}_{jk}$'s properly (see (2.7)) so that an exact computation occurs for the term $\sum_{j,k=0}^{1}\mathcal{A}_{jk}(\partial_t^j u, \partial_t^k u)$. To realize this idea, we first switch notations to facilitate our computation. We use D' and D'' to denote the first and second entry, respectively, that is, we write

$$\mathcal{A}_{jk}(\partial_t^j u, \partial_t^k u) = \mathcal{A}_{jk}(D', D'')[\partial_t^j u][\partial_t^k u], \qquad \mathcal{B}_{jk}(\partial_t^j u, \partial_t^k u) = \mathcal{B}_{jk}(D', D'')[\partial_t^j u][\partial_t^k u].$$

Then it follows from Lebniz' rule that

$$(1-\Delta)U := \sum_{j,k=0}^{1}\big((1+|D'|^2) + (2\langle D', D''\rangle - 1) + (1+|D''|^2)\big)\mathcal{B}_{jk}(D', D'')[\partial_t^j u][\partial_t^k u], \tag{2.4}$$ {eqn:Lebniz

$$\partial_t^2 U := \sum_{j,k=0}^{1}\mathcal{B}_{jk}(D', D'')\big([\partial_t^{j+2}u][\partial_t^k u] + 2[\partial_t^{j+1}u][\partial_t^{k+1}u] + [\partial_t^j u][\partial_t^{k+2}u]\big). \tag{2.5}$$ {eqn:Lebniz

In the second equation, we reduce the number of ∂_t derivatives via the differential equations. More specifically, we plug in

$$\partial_t^{l+2}u = \partial_t^l\big(\Delta u - u - F\big) = -(|D_x|^2+1)\partial_t^l u - \partial_t^l F, \quad \forall l \in \mathbb{N},$$

with $l = j$, $l = k$ respectively for the first and third term in (2.5) while for j or k equal to 1, we also replace the second term in (2.5) with $l = 0$. Therefore, (2.5) becomes

$$\begin{aligned}\partial_t^2 U := &-\sum_{j,k=0}^{1}\mathcal{B}_{jk}(D', D'')\left((|D'|^2+1) + (|D''|^2+1)\right)[\partial_t^j u][\partial_t^k u]\\ &+ 2\mathcal{B}_{00}(D', D'')[\partial_t u][\partial_t u] - 2\mathcal{B}_{10}(D', D'')(|D'|^2+1)[u][\partial_t u]\\ &- 2\mathcal{B}_{01}(D', D'')(|D''|^2+1)[\partial_t u][u] + 2\mathcal{B}_{11}(D', D'')(|D'|^2+1)(|D''|^2+1)[u][u]\\ &- \sum_{j,k=0}^{1}\mathcal{B}_{jk}(D', D'')\big([\partial_t^j F][\partial_t^k u] + [\partial_t^j u][\partial_t^k F]\big)\\ &- 2\mathcal{B}_{10}(D', D'')[F][\partial_t u] - 2\mathcal{B}_{01}(D', D'')[\partial_t u][F] + 2\mathcal{B}_{11}(D', D'')(|D'|^2+1)[u][F]\\ &+ 2\mathcal{B}_{11}(D', D'')(|D''|^2+1)[F][u] + 2\mathcal{B}_{11}(D', D'')[F][F].\end{aligned} \tag{2.6}$$ {eqn:Lebniz

Note that the terms involving F or ∂F will be third order at $u = 0$ so those can be considered as good terms. Apart from those, we expect an exact cancellation. To see this, we collect the terms in $(\Box - 1)U$ that do not involve f :

$$\begin{aligned}&-2\mathcal{B}_{00}(D', D'')[\partial_t u][\partial_t u] + 2\mathcal{B}_{10}(D', D'')(|D'|^2+1)[u][\partial_t u] + 2\mathcal{B}_{01}(D', D'')(|D''|^2+1)[\partial_t u][u]\\ &-2\mathcal{B}_{11}(D', D'')(|D'|^2+1)(|D''|^2+1)[u][u] - \sum_{j,k=0}^{1}\mathcal{B}_{jk}(D', D'')\left(2\langle D', D''\rangle - 1\right)[\partial_t^j u][\partial_t^k u].\end{aligned}$$

We enforce the conditions that this is equal to the sum in (2.1) and this leads to

$$\begin{aligned}\mathcal{B}_{00}(\xi,\eta)(2\langle\xi,\eta\rangle-1)+2\mathcal{B}_{11}(\xi,\eta)(1+|\xi|^2)(1+|\eta|^2)&=-\mathcal{A}_{00}(\xi,\eta),\\ \mathcal{B}_{11}(\xi,\eta)(2\langle\xi,\eta\rangle-1)+2\mathcal{B}_{00}(\xi,\eta)&=-\mathcal{A}_{11}(\xi,\eta),\\ \mathcal{B}_{01}(\xi,\eta)(2\langle\xi,\eta\rangle-1)-2\mathcal{B}_{10}(\xi,\eta)(1+|\xi|^2)&=-\mathcal{A}_{01}(\xi,\eta),\\ \mathcal{B}_{10}(\xi,\eta)(2\langle\xi,\eta\rangle-1)-2\mathcal{B}_{01}(\xi,\eta)(1+|\eta|^2)&=-\mathcal{A}_{10}(\xi,\eta).\end{aligned}$$

The solution is then given by

:defn_B_jk}

$$\begin{aligned}\mathcal{B}_{00}(\xi,\eta)&=\big((2\langle\xi,\eta\rangle-1)\mathcal{A}_{00}(\xi,\eta)-2(1+|\xi|^2)(1+|\eta|^2)\mathcal{A}_{11}(\xi,\eta)\big)\mathcal{K}(\xi,\eta),\\ \mathcal{B}_{11}(\xi,\eta)&=\big((2\langle\xi,\eta\rangle-1)\mathcal{A}_{11}(\xi,\eta)-2\mathcal{A}_{00}(\xi,\eta)\big)\mathcal{K}(\xi,\eta),\\ \mathcal{B}_{01}(\xi,\eta)&=\big((2\langle\xi,\eta\rangle-1)\mathcal{A}_{01}(\xi,\eta)-2(1+|\xi|^2)\mathcal{A}_{10}(\xi,\eta)\big)\mathcal{K}(\xi,\eta),\\ \mathcal{B}_{10}(\xi,\eta)&=\big((2\langle\xi,\eta\rangle-1)\mathcal{A}_{10}(\xi,\eta)-2(1+|\eta|^2)\mathcal{A}_{01}(\xi,\eta)\big)\mathcal{K}(\xi,\eta),\end{aligned}\tag{2.7}$$

where

of_calB_op}

$$\mathcal{K}(\xi,\eta):=\big(4(1+|\xi|^2)(1+|\eta|^2)-(2\langle\xi,\eta\rangle-1)^2\big)^{-1}.\tag{2.8}$$

Remark 2.1. Note that the fact that this is the reciprocal of some determinant. This invertibility is called the non-resonant condition. Note that for wave equations, the $\mathcal{K}$ we obtain here would be 0 when ξ and η are along the same direction. However, if we have the null condition, then the division problem will become zero times something is equal to zero, circumventing the issue of dividing by zero.

Remark 2.2. If one compares with the exposition in [Hör97], we slightly vary the convention of $\mathcal{A}_{jk}$ and the Klein-Gordon operator in this note.

Note that $\mathcal{B}_{jk}(D',D'')$ differs from $\mathcal{K}$ only by a differential operator with real coefficients, of order $\leqslant 3-j$ when acting on the first argument and order $\leqslant 3-k$ when acting on the second argument. Therefore, with the price of loss of derivatives, it suffices to show desired mapping properties of $\mathcal{K}$. The following lemma basically tells us that with the price of loss of derivatives, the bilinear operator $\mathcal{K}$ can be viewed as products regarding the estimates we wish to have.

Lemma 2.3. *For $1\leqslant p,q,r\leqslant\infty$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, then*

$$\|\mathcal{K}(D',D'')[u][v]\|_{L^r}\lesssim\sum_{|\alpha|\leqslant\nu}\|\partial^\alpha u\|_{L^p}\sum_{|\beta|\leqslant\nu}\|\partial^\beta v\|_{L^q},\quad u,v\in\mathscr{S}(\mathbb{R}^n),$$

if ν is the smallest integer $>\frac{n-4}{2}$ when $n>3$, $\nu=1$ when $n=3$ and $\nu=0$ when $n=1,2$.

Proof. For $n\geqslant 2$, we write

$$\mathcal{K}(\xi,\eta)=\langle\xi\rangle^{2\nu}\langle\eta\rangle^{2\nu}\langle\xi\rangle^{-2\nu}\langle\eta\rangle^{-2\nu}\mathcal{K}(\xi,\eta)=\sum_{|\alpha|,|\beta|\leqslant\nu}c_{\alpha\beta}\xi^\alpha\eta^\beta\mathcal{K}_{\alpha\beta}(\xi,\eta),$$

where $c_{\alpha\beta}$ are constants and

$$\mathcal{K}_{\alpha\beta}(\xi,\eta)=\xi^\alpha\eta^\beta\langle\xi\rangle^{-2\nu}\langle\eta\rangle^{-2\nu}\mathcal{K}(\xi,\eta).$$

Note that

$$\mathcal{K}(D', D'')[u][v] = \sum_{|\alpha|,|\beta|\leqslant\nu} c_{\alpha\beta}\mathcal{K}_{\alpha\beta}(D', D'')[D^\alpha u][D^\beta u],$$

the desired estimate would follow from Lemma 1.8 and 1.9 if one could prove $\partial_\xi^\delta\partial_\eta^\gamma\mathcal{K}_{\alpha\beta}(\xi,\eta) \in L^2(\mathbb{R}^{2n})$ for all $|\delta|, |\gamma| \leqslant \frac{n+2}{2}$. In fact, one can show that $\partial_\xi^\delta\partial_\eta^\gamma\mathcal{K}_{\alpha\beta}(\xi,\eta) \in L^2(\mathbb{R}^{2n})$ for all δ, γ. See [Hör97, Lemma 7.8.5]. We only mention that this statement is heuristically correct given the decay rate of $\mathcal{K}$ at infinity.

For $n = 1$, we need to show that $\check{\mathcal{K}} \in L^1$ directly. Note that

$$\mathcal{K}(\xi,\eta) = \left(4(\xi^2+\eta^2+\xi\eta)+3\right)^{-1}, \quad \xi,\eta \in \mathbb{R}.$$

This corresponds to the fundamental solution of the operator $-4(\partial_x^2+\partial_y^2+\partial_x\partial_y)+3$ in $(x, y) \in \mathbb{R}^2$. By a linear transformation, it suffices to find for $-\Delta_{\mathbb{R}^2}+1$. The inverse Fourier transform can be computed via the identity

$$\frac{1}{1+|\xi|^2}e^{-\varepsilon(1+|\xi|^2)} = \int_\varepsilon^\infty e^{-\lambda(1+|\xi|^2)}\,d\lambda$$

and the theory of Bessel functions. It has logarithmic singularity at origin and exponentially decay at infinity. (See also [MTT97].) By making a holomorphic extension to a strip, one can also observe the exponential decay property from Paley-Wiener theorem directly. □

Now we are ready to state and prove the main theorem.

Theorem 2.4. *Assume that the function F in (0.2) is in C^∞ and vanishes of second order at 0, and the dimension $n \geqslant 3$. Then (0.2) has a global solution $C^\infty([0,\infty)\times\mathbb{R}^n)$ with Cauchy data*

$$u(0) = \varepsilon u_0, \quad \partial_t u(0) = \varepsilon u_1, \quad u_0, u_1 \in \mathscr{S}(\mathbb{R}^n)$$

for ε sufficiently small.

Proof. We assume the same bootstrap conditions with $s \geqslant 3n+11$ to compensate the loss of derivatives.

We write $u = U + V$ with U given by (2.2) with $\mathcal{B}_{jk}$ given in (2.7). Since

$$\partial^\gamma U = \sum_{\gamma'+\gamma''=\gamma}\frac{\gamma!}{\gamma'!\gamma''!}\sum_{j,k=0}^{1}\mathcal{B}_{jk}(D', D'')[\partial^{\gamma'}\partial_t^j u][\partial^{\gamma''}\partial_t^k u],$$

we could estimate L^∞ norm of $\partial^\gamma U$ by one copy of (BA-1-KG)(Sobolev embedding) and one copy of (BA-2-KG) when $|\gamma| \leqslant \frac{s}{2}+2$ by our choice of s.

For $\partial^\gamma V$ with $|\gamma| \leqslant \frac{s}{2}+2$, it suffices to estimate the source term of (2.3) in L^1 norm like in (1.6) as we know that our choice of $\mathcal{B}_{jk}$ ensures that the source term of (2.3) vanishes of third order at 0. □

3. Hyperboloidal method

One limitation in this hyperboloidal method is that we need to assume the initial data is compactly supported. Suppose $\operatorname{supp} u_j \subset \{x : |x| \leqslant B\}$ $(j = 0, 1)$. Without loss of generality, we shift $2B$ in time, that is, consider the following initial value problem

$$\Box u - u = f,\ t \geqslant 2B, \quad u(2B, \cdot) = \varepsilon u_0, \quad \partial_t u(2B, \cdot) = \varepsilon u_1. \tag{3.1}$$ {eqn:IVP_at

Going forward, we always assume that $t \geqslant 2B$.

It follows that if the implication

$$(t, x) \in \operatorname{supp} f,\ t \geqslant 2B \Rightarrow |x| \leqslant t - B$$

holds, then

erboloidal}
$$(t, x) \in \operatorname{supp} u,\ t \geqslant 2B \Rightarrow |x| \leqslant t - B. \tag{3.2}$$

For our purpose that $f = F(u, u', u'')$, from the support condition of initial data and the support condition of inductive approximate solution in the local existence theorem, we know that u has the desired support condition (3.2). Moreover, this tells us that

$$t^2 - |x|^2 \geqslant B(t + |x|) \geqslant 2B^2, \quad \forall (t, x) \in \operatorname{supp} u, \quad t \geqslant 2B.$$

In view of the support condition, the local existence theorem and the geometry of hyperboloids, by choosing ε small enough, we could assume C^∞ initial data on $\{(t, x) : t^2 - |x|^2 = (2B)^2\}$. (Indeed, $t^2 - |x|^2 = (2B)^2$ has intersection with $t - |x| = B$ at some finite t. This region is named as G_{2B} later.) Hence, to prove global existence theorem, we expect to set up some bootstrap argument along constant hyperboloidal foliations to conclude global existence.

We start with energy estimates on hyperboloid. We define

$$H_T := \{(t, x) : t^2 - |x|^2 = T^2\}, \quad G_T := \{(t, x) : t \geqslant 2B, \quad t^2 - |x|^2 \leqslant T^2\}.$$

Then multiplying (3.1) by $2\partial_t u$, we obtain

$$2f\partial_t u = -\partial_t(|\nabla_{t,x} u|^2 + |u|^2) + 2\nabla_x \cdot (\partial_t u \nabla_x u).$$

With H_T parametrized by x (i.e. $\mathrm{Vol}_{H_T} = dx$), the corresponding normal on H_T is given by $(1, -\frac{x}{t})$. Therefore, integrating the differential identity in the region G_T gives

ed_E_on_HT}
$$E(T; u) := \int_{H_T} \left(|\nabla_{t,x} u|^2 + 2\partial_t u \langle \nabla_x u, \frac{x}{t} \rangle + |u|^2 \right) dx = -\int_{G_T} 2f\partial_t u \, dx \, dt + E_0(2B; u), \tag{3.3}$$

where $E_0(T; u) = \sum_{|\alpha| \leqslant 1} \|\partial^\alpha u(T, \cdot)\|_{L^2}^2$ is the standard energy associated with the Klein-Gordon equation used in Lemma 1.1. $E(T; u)$ is coercive since H_T is spacelike. Completing squares leads to

are_E_T_KG}
$$\left| \nabla_x u + \partial_t u \frac{x}{t} \right|^2 + (\partial_t u)^2 \frac{\varrho^2}{t^2} + |u|^2, \quad \varrho^2 = t^2 - |x|^2. \tag{3.4}$$

In order to differentiate (3.3), we write

$$\partial_T H(\sqrt{T^2 + |x|^2} - t) = T(T^2 + |x|^2)^{-\frac{1}{2}} \delta((T^2 + |x|^2)^{\frac{1}{2}} - t),$$

where H denotes the Heaviside function. Therefore,

$$\frac{d}{dT}E(T;u) = -2\int_{H_T} T(T^2+|x|^2)^{-\frac{1}{2}} f\partial_t u\,dx \leqslant 2\left(\int_{H_T} f^2\,dx\right)^{\frac{1}{2}} E(T)^{\frac{1}{2}}$$

thanks to (3.4). Integrating in T gives

$$E(T;u)^{\frac{1}{2}} \leqslant E(2B;u)^{\frac{1}{2}} + \int_{2B}^{T}\left(\int_{H_s} f^2\,dx\right)^{\frac{1}{2}} ds. \tag{3.5}$$ {eqn:E_T_es

In $G_{2B}\cap \operatorname{supp} u$, we have $t \leqslant 5B/2$, so (3.3) and the standard energy estimate in Lemma 1.1 implies that

$$\begin{aligned} E(2B;u) &\leqslant \int_{2B}^{\frac{5B}{2}} \|f(t,\cdot)\|_{L^2_x} E_0(t;u)^{\frac{1}{2}}\,dt + E_0(2B;u) \\ &\leqslant \left(E_0(2B;u)^{\frac{1}{2}} + \int_{2B}^{\frac{5B}{2}} \|f(s,\cdot)\|_{L^2_x}\,ds\right)\int_{2B}^{\frac{5B}{2}} \|f(t,\cdot)\|_{L^2_x}\,dt + E_0(2B;u) \\ &\leqslant 2\left(E_0(2B;u)^{\frac{1}{2}} + \int_{2B}^{\frac{5B}{2}} \|f(t,\cdot)\|_{L^2_x}\,dt\right)^2. \end{aligned}$$

Plugging into (3.5), we obtain good control of $E(T;u)$. In what follows, we ignore the simple and standard estimate in G_{2B} and will be happy with estimates adapted to hyperboloids only like in (3.5). We also need to set up energy estimates for a linear perturbation of Klein-Gordon equation. We summarize the results below.

{lemma_lin_

Lemma 3.1. *Suppose u is a solution to*

$$\Box u - u + \sum_{\mu,\nu=0}^{n} \gamma^{\mu\nu}(t,x)\partial_\mu\partial_\nu u + \sum_{\nu=0}^{n}\gamma^{\nu}(t,x)\partial_\nu u = f$$

with the support condition (3.2). If γ satisfies the following smallness conditions

$$\sup_{(t,x)\in\operatorname{supp} u} \sum_{\mu,\nu=0}^{n} |\gamma^{\mu\nu}(t,x)|\frac{t^2}{\varrho^2} \leqslant c_n,$$

$$\sum_{\mu,\nu,\delta=0}^{n} |\partial_\mu\gamma^{\nu\delta}(t,x)| + \sum_{\nu=0}^{n}|\gamma^{\nu}(t,x)| \leqslant \lambda t^{-\kappa},$$

where c_n is a positive constant and $\kappa \geqslant 1$, then

$$E(T;u)^{\frac{1}{2}} \leqslant \left(3E(2B;u)^{\frac{1}{2}} + 4\int_{2B}^{T}\left(\int_{H_s} f^2\,dx\right)^{\frac{1}{2}} ds\right)\exp\left(\int_{2B}^{T} C_n\lambda s^{-\kappa}\right).$$

Now we are ready to introduce a Sobolev embedding adapted to hyperboloids. A stronger version with dyadic sum in time involved can be found in [Hör97, Section 7.3].

{lemma_Sobo

Lemma 3.2. *If $\nu > \frac{n}{2}$ is an integer, then for $u \in C^\nu$, $T \geqslant 2B$, we have*

$$\sup_{H_T} t^n |u(t,x)|^2 \lesssim \sum_{|I| \leqslant \nu} \int_{H_T} |Z^I u|^2 \, dx, \tag{3.6}$$ {eqn:Sobole

where Z^I denotes any product of the vector fields ∂_{x^μ} with $\mu = 0, 1, \cdots, n$ and

$$Z_{0j} = t\partial_j + x^j \partial_t, \quad Z_{jk} = x^j \partial_k - x^k \partial_j, \quad j, k = 1, \cdots, n.$$

Proof. Since H_T is parametrized by x, we consider the coordinate derivative ∂_j applied to u on H_T :

$$\partial_j \left(u(\sqrt{T^2 + |x|^2}, x) \right) = \left(\frac{x^j}{\sqrt{T^2 + |x|^2}} \partial_t u + \partial_j u \right) (\sqrt{T^2 + |x|^2}, x) = \frac{1}{t}(x^j \partial_t + t\partial_j) u \Big|_{H_T}.$$

Therefore, it suffices to prove

nterm_step} $$\sup_{x \in \mathbb{R}^n} (T^2 + |x|^2)^{\frac{n}{2}} |U(x)|^2 \lesssim \sum_{|\alpha| \leqslant \nu} \int (T^2 + |x|^2)^{|\alpha|} |\partial_x^\alpha U(x)|^2 \, dx, \quad U(x) = u(\sqrt{T^2 + |x|^2}, x). \tag{3.7}$$

This actually just follows from the standard Sobolev embedding lemma

$$\sup_{x \in B_\lambda} |U(x)|^2 \lesssim \sum_{|\alpha| \leqslant \nu} \lambda^{2|\alpha| - n} \int_{B_\lambda} |\partial_x^\alpha U(x)|^2 \, dx$$

with constants independent of the domain. Indeed, for any $x_0 \in \mathbb{R}^n$, denote $t_0 := \sqrt{T^2 + |x_0|^2}$. We consider the ball $B(x_0, \frac{1}{2} t_0)$, then

$$t_0^n |U(x_0)|^2 \lesssim \sum_{|\alpha| \leqslant \nu} t_0^{2|\alpha|} \int_{B(x_0, \frac{1}{2} t_0)} |\partial_x^\alpha U(x)|^2 \, dx \lesssim \sum_{|\alpha| \leqslant \nu} \int_{B(x_0, \frac{1}{2} t_0)} (T^2 + |x|^2)^{|\alpha|} |\partial_x^\alpha U(x)|^2 \, dx,$$

where we use the geometric observation

$$\begin{aligned} |x - x_0| \leqslant \frac{1}{2} t_0 \Rightarrow \sqrt{T^2 + |x|^2} &\geqslant \sqrt{T^2 + ||x_0| - |x - x_0||^2} \geqslant \sqrt{t_0^2 + |x - x_0|^2 - 2|x_0||x - x_0|} \\ &\geqslant \sqrt{t_0^2 + |x - x_0|^2 - 2t_0|x - x_0|} = t_0 - |x - x_0| \geqslant \frac{1}{2} t_0. \end{aligned}$$

□

Now we state and sketch the proof of our main theorem.

Theorem 3.3. *Assume that the function F in (0.2) is in C^∞ and vanishes of second order at 0, and the dimension $n \geqslant 3$. Then (0.2) has a unique global solution $C^\infty([2B, \infty) \times \mathbb{R}^n)$ with Cauchy data*

$$u(t = 2B) = \varepsilon u_0, \quad \partial_t u(t = 2B) = \varepsilon u_1, \quad u_0, u_1 \in C_0^\infty, \quad \operatorname{supp} u_j \subset \{|x| \leqslant B\}$$

for ε sufficiently small.

Proof. By local existence theorem and the geometry of G_{2B}, we know that by choosing ε small enough, local existence theorem ensures that there is a C^∞ solution which is $O(\varepsilon)$ with all derivatives when $\varrho \leqslant 2B$. Therefore, we could start with H_{2B}. We set

$$M_s(t) := \sum_{|I|\leqslant s} E(t; Z^I u)^{\frac{1}{2}}$$

for some $s > n+4$. Then we make the bootstrap assumption that if the solution exists for $\varrho < T$, then

$$M_s(t) \leqslant 2M\varepsilon, \quad 2B \leqslant t < T. \tag{BA-1′-KG}$$ {eqn:BA-1'_

From Sobolev embedding (3.6), one can recover an analogue of (BA-2-KG) :

$$\sum_{|I|\leqslant s-\frac{n}{2}} \sup_{H_\tau} t^{\frac{n}{2}} |Z^I(t,x)| \lesssim M_s(\tau). \tag{3.8}$$ {eqn:BA-2'_

However, we only need to close (BA-1′-KG) so that it is natural to expect a proof similar to the one for Theorem 1.3 will work. Commuting (0.2) with Z^I gives

$$(\Box + \gamma^{\mu\nu}\partial_\mu\partial_\nu - 1)Z^I u + \sum_\nu \sum_J \gamma^\nu_{I,J}\partial_\nu Z^J u = f_I,$$

where $\gamma^\nu_{I,J} = 0$ unless $|I| = |J| = s$. We isolate one more term compared to (1.2) which captures all the term with $s+1$ order of derivatives. The form of such term is due to the fact that $[Z, \partial_\nu]$ is either 0 or $\pm\partial_\mu$ for some μ. This tells us that f_I only contains s derivatives in total involving Z's.

Since $s - \frac{n}{2} \geqslant 3$, it follows from (3.8) that

$$\sum |\gamma^{\mu\nu}(t,x)| + \sum |\partial_\mu \gamma^{\nu\delta}(t,x)| + \sum_\nu \sum_{|I|=|J|=s} |\gamma^\nu_{I,J}(t,x)| \lesssim M\varepsilon t^{-\frac{n}{2}}.$$

Thus Lemma 3.1 can be applied to obtain good estimates of $E(T; Z^I u)^{\frac{1}{2}}$. Similar to (1.4), we could estimate

$$\int_{2B}^t \left(\int_{H_\tau} |f_I|^2 \, dx \right)^{\frac{1}{2}} d\tau \leqslant \int_{2B}^t CM\varepsilon \tau^{-\frac{n}{2}} M_s(\tau) \, d\tau \lesssim \varepsilon \int_{2B}^t (1+\tau)^{-\frac{n}{2}} \, d\tau \lesssim \varepsilon.$$

Plugging this into Lemma 3.1 gives the improvement of (BA-1′-KG). □

Remark 3.4. In summary, these two methods take two different approaches to remedy the proof of Theorem 1.3. The normal form transformation modifies the nonlinearity so that the idea of proof can go through. On the other hand, the hyperboloidal method modifies the bootstrap assumptions by deleting the analogue of (BA-2-KG) on hyperboloids. Without the need to close this bootstrap, the second order vanishing assumption is enough. (Recall Remark 1.4.) However, to prove the analogue of (BA-1-KG), one needs to use some uniform decay estimates like (BA-2-KG). This can be obtained for free from the Sobolev embedding adapted to hyperboloids See Lemma 3.2. One should also compare this with the classical Klainerman-Sobolev inequality

$$(1+t+|x|)^{n-1}(1+|t-|x||)|u(t,x)|^2 \lesssim \sum_{|I|\leqslant \frac{n+2}{2}} \|Z^I u(t,\cdot)\|^2,$$

where $u \in \mathscr{S}$ in $(t-1,t+1)\times\mathbb{R}^n$. Opposed to this $t^{-\frac{n-1}{2}}$ decay rate, we could obtain $t^{-\frac{n}{2}}$ decay rate from the one adapted to hyperboloids, recovering the decay from fundamental solutions of Klein-Gordon equations. Note that this is in particular crucial when one discusses $n = 3$ case as we need the integrability condition of $\int t^{-\frac{n}{2}}\,dt$. On the other hand, when $n = 4$, small data global wellposedness for semilinear wave equations is established using Klainerman-Sobolev inequality by exactly the same idea. Moreover, when $n = 3$, we need to assume null conditions.

Remark 3.5. It might be tempting to say that this hyperboloidal method can be adapted to the wave equations when $n = 3$ without null conditions. In this case, since we do not have zeroth order control in $E(T;u)$ anymore, even if the energy is still coercive, it does not give full control of $\dot{H}^1$ norm. Instead, it only gives the control of derivatives tangential to H_T. Then it makes sense that tangential derivatives behave better than $t^{-\frac{n-1}{2}}$. For M_s, it only contains control of $|\alpha|$-order derivatives with $|\alpha| \leqslant s+1$, where at least one derivative is along tangential direction.

Remark 3.6. The better decay of Klein-Gordon equations on Minkowski background compared to wave equations does not give too much helpful information for Klein-Gordon on black hole backgrounds. For Klein-Gordon equations, it is important that the mass term has the correct sign here to generate positive definite energies. However, on black hole spacetimes, there might be an attracting Coloumb potential involved, which will cause the trapping not being unstable anymore. Also, the existence of quasinormal modes will contribute to non-decaying phenomenon as well.

Remark 3.7. [Kla87] also introduces a conjugation $v = \varrho^{\frac{n}{2}}u$. The idea here is to look at the fundamental solution. A crude analysis via stationary phase gives something like $t^{-\frac{n}{2}}e^{i\sqrt{t^2-|x|^2}}$ for x bounded. However, since delta distribution is Lorentzian invariant, we expect the fundamental solution to have such property as well. So one can expect $t^{-\frac{n}{2}}$ is actually corresponding to $\varrho^{-\frac{n}{2}}$. This is why we want to multiply the solution by $\varrho^{\frac{n}{2}}$. In addition to choosing x as the parametrization, [Kla87] works with ϱ.

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