NONLINEAR STABILITY OF EXTREMAL RN IN SPHERICAL SYMMETRY

NING TANG

This note is intended as an overview of [AKU24].

1. Preliminaries

1.1. **Einstein-Maxwell system in the spherical symmetry.** The *Einstein–Maxwell-neutral scalar field system* consists of a spacetime $(\mathcal{M}, g)$ equipped with an electromagnetic field F and a scalar field $\phi : \mathcal{M} \to \mathbb{R}$ satisfying the equations

$$\mathrm{Ric}(g) - \tfrac{1}{2}Rg = 2(T^{\mathrm{EM}} + T^{\mathrm{SF}}), \tag{1.1}$$

$$dF = 0, \quad d \star F = 0, \tag{1.2}$$

$$\Box_g \phi = 0, \tag{1.3}$$

where the energy-momentum tensors are defined by

$$T^{\mathrm{EM}}_{\mu\nu} := F_{\mu\alpha}F^{\alpha}{}_{\nu} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$
$$T^{\mathrm{SF}}_{\mu\nu} := \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi.$$

We say that $(\mathcal{M}, g, F, \phi)$ is *spherically symmetric* if $(\mathcal{M}, g)$ is spherically symmetric, F has the form (1.12), and ϕ is independent of the angular coordinates. In this case, Einstein's equation (1.1) reduces to the wave equations

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r\partial_v r}{r} + \frac{\Omega^2 e^2}{4r^3}, \tag{1.4}$$
$$\partial_u\partial_v \log\Omega^2 = \frac{\Omega^2}{2r^2} + \frac{2\partial_u r\partial_v r}{r^2} - \frac{\Omega^2 e^2}{r^4} - 2\partial_u\phi\partial_v\phi,$$

and Raychaudhuri's equations

$$\partial_u\left(\frac{\partial_u r}{\Omega^2}\right) = -\frac{r}{\Omega^2}(\partial_u\phi)^2, \tag{1.5}$$

$$\partial_v\left(\frac{\partial_v r}{\Omega^2}\right) = -\frac{r}{\Omega^2}(\partial_v\phi)^2. \tag{1.6}$$

The Maxwell equations (1.2) are automatically satisfied since e is constant. Finally, the wave equation (1.3) is equivalent to

$$\partial_u\partial_v\phi = -\frac{\partial_v r\partial_u\phi}{r} - \frac{\partial_u r\partial_v\phi}{r}. \tag{1.7}$$

We recall the shorthand notations for spherically symmetric Einstein equations. The renormalized Hawking mass is

$$\varpi := m + \frac{e^2}{2r}$$

and the mass aspect function

$$\mu := \frac{2m}{r} = \frac{2\varpi}{r} - \frac{e^2}{r^2}, \tag{1.8}$$

The traditional notations are as follows :

$$\nu := \partial_u r, \quad \lambda := \partial_v r.$$

Since

$$m = \frac{r}{2}\left(1 + \frac{4\partial_u r \partial_v r}{\Omega^2}\right). \tag{1.9}$$

we have

$$\kappa := -\frac{\Omega^2}{4\partial_u r} = \frac{\lambda}{1-\mu}, \quad \gamma := -\frac{\Omega^2}{4\partial_v r} = \frac{\nu}{1-\mu}.$$

We can derive the fundamental relations

$$\partial_u \varpi = (1-\mu)\frac{r^2}{2\nu}(\partial_u \phi)^2, \tag{1.10}$$

$$\partial_v \varpi = \frac{r^2}{2\kappa}(\partial_v \phi)^2, \tag{1.11}$$

$$\partial_u \kappa = \frac{r\kappa}{\nu}(\partial_u \phi)^2,$$

$$\partial_v \gamma = \frac{r\gamma}{\lambda}(\partial_v \phi)^2.$$

1.2. **Geometry of Reissner–Nordström family.** The Reissner–Nordström family forms the unique family of spherically symmetric asymptotically flat four-dimensional solutions to the Einstein–Maxwell equations. (Note that there is $\mathrm{AdS}_2 \times \mathbb{S}^2$ if eliminating the asymptotically flat assumption. See [Ton].) It is of the form

$$g_{M,e} = -D\,dt^2 + D^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin\theta^2\,d\varphi^2),$$

with a one-form $A = -\frac{Q}{r}\,dt - B\cos\theta\,d\varphi$, where $D(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$ and $|e| = \sqrt{Q^2+B^2}$.

The extremal case refers to the case $|e| = M$, where $D(r) = (1 - \frac{M}{r})^2$. Here, e is a real parameter representing the charge of the electromagnetic field. Restricting to spherically symmetric electromagnetic fields with constant Coulomb charge $e \in \mathbb{R}$, we have

$$F = dA = \frac{e}{r^2}\,dr \wedge\, dt.$$

In order to put Reissner-Nordström metric in double null gauge, we define the retarded time coordinate $u := \frac{1}{2}(t - r_*)$ and the advanced time coordinate $v := \frac{1}{2}(t + r_*)$. Here, the tortoise coordinate

$$r_*(r) := \int^r \frac{dr'}{D(r')} = r - M - \frac{M^2}{r-M} + 2M\log(r-M) + C$$

in the extremal case. In double null gauge, the metric takes the Eddington-Finkelstein double null form

$$g_{M,e} = -4D\,du\,dv + r^2 g_{\mathbb{S}^2}$$

and hence $\Omega^2 = 4D$. Then $du \wedge dv = \frac{1}{2}(dt \wedge dr_*) = \frac{1}{2}D^{-1}\,dt \wedge dr$. Thus,

$$F = -\frac{\Omega^2 e}{2r^2}\,du \wedge\, dv. \tag{1.12}$$

From the identity $r_* = v - u$, we know that in double null coordinates, $\nu = \partial_u r = -D$ and $\lambda = \partial_v r = D$. In $g_{M,e}$, we have $\varpi = M$ and hence

$$\mu = \frac{2M}{r} - \frac{e^2}{r^2} = 1 - D.$$

We also derive that

$$\kappa = \frac{\lambda}{1-\mu} = 1, \quad \gamma := \frac{\nu}{1-\mu} = -1. \tag{1.13}$$

Notice that we have the freedom to choose r at a given point $(u_0, v_0) \in \mathbb{R}^2$ by simply changing the origin of the (u, v)-coordinates. That is, given parameters M and $|e| \leq M$, for any $R_0 \in (r_+, \infty)$, we can uniquely define r such that $r(u_0, v_0) = R_0$ such that the geometric quantities defined above make it a Reissner-Nordström black hole exterior.

Note that the double null coordinates only cover the domain of outer communication if $|e| \leq M$. The event horizon $\mathcal{H}^+$ formally corresponds to $u = +\infty$ and null infinity $\mathcal{I}^+$ formally corresponds to $v = +\infty$. With this understanding, we may extend geometric quantities to the horizon by noting that $D(\infty, \cdot) = D|_{\mathcal{H}^+} = 0$.

To actually extend past the event horizon, we examine in the ingoing Eddington-Finkelstein form

$$g_{M,e} = -4D\, dv^2 + 4\, dv\, dr + r^2 g_{\mathbb{S}^2},$$

which is regular across $\mathcal{H}^+$. In (v, r) coordinates, we define $T := \partial_v$, which is the time-translation Killing vector field. We remark that this is exactly ∂_t in the (t, r)-coordinates in the domain of outer communication. We also define $Y := \partial_r$ in these coordinates. Since $g(\nabla_{\partial_v}\partial_v, \partial_v) = \frac{1}{2}\partial_v(-4D) = 0$ and $g(\nabla_{\partial_v}\partial_v, \partial_r) = -g(\partial_v, \nabla_{\partial_r}\partial_v) = 2\partial_r D$. Thus,

$$\nabla_T T = \frac{1}{2}\partial_r D \partial_v + (\frac{1}{2}D\partial_r D)\partial_r.$$

Restricting to $\mathcal{H}^+$ (i.e., $r = r_+ = M + \sqrt{M^2 - e^2}$), we obtain

$$\nabla_T T|_{\mathcal{H}^+} = \frac{1}{2}\frac{r_+ - r_-}{r_+^2} T|_{\mathcal{H}^+} =: \varkappa T|_{\mathcal{H}^+}.$$

One can see the redshift factor (surface gravity) $\varkappa = 0$ in the extremal case.

In a general setup,

$$\varkappa := \frac{1}{r^2}(\varpi - \frac{e^2}{r}).$$

See for instance [Daf05a, Section 6].

To understand the causality of these vector fields better, we go back to the double null coordinates. One can check that

$$Y = \frac{1}{\partial_u r}\partial_u, \quad T = \frac{4\lambda}{\Omega^2}\partial_u - \frac{4\nu}{\Omega^2}\partial_v = \partial_u + \partial_v,$$

where $T = \frac{4\lambda}{\Omega^2}\partial_u - \frac{4\nu}{\Omega^2}\partial_v$ is the Kodama vector field in a generic double null coordinate system. Therefore, one can see that T is future-directed timelike for $r > r_+$ while Y is past-directed null and is transverse to $\mathcal{H}^+$.

1.3. **The Couch–Torrence conformal inversion.** ERN has a discrete conformal symmetry Φ discovered by Couch and Torrence [CT84], where a nice exposition can be found in [Are18].

In (t, r_*) coordinates,

$$\Phi(t, r_*, \theta, \varphi) = (t, -r_*, \theta, \varphi).$$

Recall that $r_* = 0$ precisely on the photon sphere, so this symmetry fixes the photon sphere. It sends $\mathcal{H}^+$ to $\mathcal{I}^+$ and vice versa.

In standard coordinates (t, r), the Couch-Torrence inversion takes the form $\Phi(t, r) = (t, r')$, where $r'(r) = \frac{rM}{r-M}$. Moreover, in double null coordinates, $\Phi(u, v) = (v, u)$.

Therefore, one can see the relation

$$r^2\partial_v \mapsto (r')^2\partial_u = \frac{r^2 M^2}{(r-M)^2}\partial_u = M^2\frac{1}{D}\partial_u = M^2\frac{1}{\partial_u r}\partial_u = M^2 Y,$$

where we recall that $Y = \partial_r$ in the (v, r) coordinates. This gives the relation between Newman-Penrose constant I_0 and the horizon charge H_0.

1.4. **Aretakis instability.** Consider the linear wave equation on a fixed background $g : \Box_g \phi = 0$ in spherical symmetry, i.e., the form (1.7). We introduce the radiation field $\psi := r\phi$, then

$$\partial_u \partial_v \psi = (\partial_u \partial_v r)\phi.$$

Recall from (1.4) in Einstein equations that

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + \frac{\Omega^2 e^2}{4r^3} = \frac{\kappa \partial_u r}{r} - \frac{\partial_u r(1-\mu)\kappa}{r} - \frac{\kappa e^2 \partial_u r}{r^3} = \kappa\nu(\frac{\mu}{r} - \frac{e^2}{r^3}) = 2\kappa\nu r^{-2}(m - \frac{e^2}{2r}) = 2\kappa\nu\varkappa.$$

Now if we specialize to $g = g_{ERN}$, then $\kappa = 1$, $\nu = -D$ and $2\varkappa = D'$. Hence, $\partial_v \partial_u r = -DD'$ and $\Box_{g_{ERN}} \phi = 0$ reads as follows :

$$\partial_u \partial_v \psi = -DD' \frac{\psi}{r}.$$

Moreover, we compute

$$\partial_v (Y\psi) = \partial_v (\frac{\partial_u \psi}{\partial_u r}) = \frac{\partial_v \partial_u \psi}{\partial_u r} - \frac{\partial_u \psi}{(\partial_u r)^2} \partial_v \partial_u r = -D' \frac{\psi}{r} + D' Y\psi.$$

Since $D'|_{\mathcal{H}^+} = 0$ in the extremal case, we know that $Y\psi$ is a constant along $\mathcal{H}^+$. Therefore, in sharp contrast to the subextremal case, $Y\psi$ does not decay along $\mathcal{H}^+$. This constant is written as $H_0[\phi]$ and is called the zeroth Aretakis charge of ϕ.

Since $[\partial_v, Y] = -\frac{\partial_v \partial_u r}{(\partial_u r)^2} \partial_u = D'Y$, we commute Y again and arrive at

$$\partial_v (Y^2 \psi) = [\partial_v, Y](Y\psi) + Y(\partial_v Y \psi) = D' Y^2 \psi + Y(-D' \frac{\psi}{r} + D' Y \psi).$$

Since $YD' = 2M \frac{1}{\partial_u r} \partial_u ((1 - M/r) r^{-2}) = 2M^2 r^{-4} + 2M(1 - M/r) \frac{\partial_u (r^{-2})}{\partial_u r}$, we know that $YD'|_{\mathcal{H}^+} = 2M^{-2}$ and hence

$$\partial_v (Y^2 \psi|_{\mathcal{H}^+}) = 2M^{-2} Y\psi - 2M^{-3} \psi = 2M^{-2} H_0[\phi] + \text{decaying terms}.$$

Here, one can heuristically use the Couch-Torrence symmetry to see that one can expect ψ decays like v^{-1} on $\mathcal{H}^+$. Therefore, integrating in v for v large, one obtains that

$$|Y^2 \psi| \gtrsim |H_0[\phi]| v \quad \text{on } \mathcal{H}^+, \quad v \gg 1.$$

In fact, one can derive that $|Y^k \psi| \gtrsim |H_0[\phi]| v^{k-1}$ on $\mathcal{H}^+$ for $k \geq 1$ and v large.

1.5. **Conformal energy and r^p-weighted vector field method.** We follow [AAG20, Section 5.2] in this part. Here, we choose $r_H \in (M, 2M)$ and $r_{\mathcal{I}} > 2M$ such that $r_*(r_{\mathcal{H}}) = -r_*(r_{\mathcal{I}})$. Then we introduce the corresponding partition of the spacetime region $\mathcal{R}$: $\mathcal{R} = \mathcal{A}^{\mathcal{H}} \cup \mathcal{B} \cup \mathcal{A}^{\mathcal{I}}$, where

$$\mathcal{A}^{\mathcal{H}} := \mathcal{R} \cap \{r \geq r_{\mathcal{H}}\}, \quad \mathcal{B} := \mathcal{R} \cap \{r_{\mathcal{H}} < r < r_{\mathcal{I}}\}, \quad \mathcal{A}^{\mathcal{I}} := \mathcal{R} \cap \{r \leq r_{\mathcal{I}}\}.$$

See Figure 1. We note that this definition is in view of the Couch-Torrence conformal inversion.

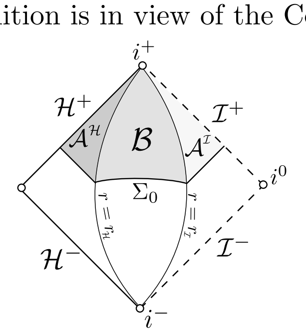

FIGURE 1. The regions $\mathcal{A}^{\mathcal{H}}, \mathcal{B}$ and $\mathcal{A}^{\mathcal{I}}$.

To obtain an analog of the r^p-hierarchy both at the near-infinity region $\mathcal{A}^{\mathcal{I}}$ and at the near-horizon region $\mathcal{A}^{\mathcal{H}}$, we examine the conformal energy. First, we define $N_\tau^{\mathcal{I}} := \Sigma_\tau \cap \mathcal{A}^{\mathcal{I}}$ and $N_\tau^{\mathcal{H}} := \Sigma_\tau \cap \mathcal{A}^{\mathcal{H}}$. See Figure 2.

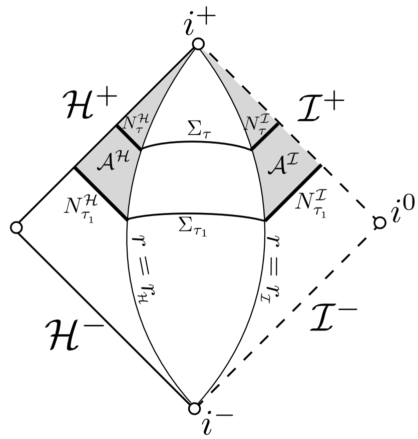

FIGURE 2. The hypersurfaces $N_\tau^{\mathcal{I}}$ and $N_\tau^{\mathcal{I}}$.

Then the conformal energies are given by

$$\textbf{Conformal energy near } \mathcal{I}^+ : \quad \mathcal{C}_{N_\tau^{\mathcal{I}}}[\psi] = \int_{N_\tau^{\mathcal{I}}} r^2 \cdot (\partial_v \psi)^2 \, d\omega dv$$

[1]

$$\textbf{Conformal energy near } \mathcal{H}^+ : \quad \mathcal{C}_{N_\tau^{\mathcal{H}}}[\psi] = \int_{N_\tau^{\mathcal{H}}} \frac{1}{-\partial_u r} (\partial_u \psi)^2 \, d\omega du.$$

Since Φ in Section 1.3 has mapping property $(u, v) \mapsto (v, u)$, $r \mapsto \frac{rM}{r-M}$, and $N_\tau^{\mathcal{I}} \mapsto N_\tau^{\mathcal{H}}$, a direct calculation implies that $\mathcal{C}_{N_\tau^{\mathcal{I}}}[\psi] \mapsto \mathcal{C}_{N_\tau^{\mathcal{H}}}[\psi]$ under Φ, where we also use $D = -\partial_u r = (1 - M/r)^2$. Moreover, one may worry about the degeneracy in the energy definition near $\mathcal{H}^+$. In fact, the degeneracy is only because of the poor choice of coordinates and we notice that

$$(\partial_u \psi)^2 (-\partial_u r)^{-1} \, du$$

is invariant under reparametrization of the double null gauge and is equal to $(Y\psi)^2 \, dr$ in the ingoing Eddington-Finkelstein coordinates (v, r), which is manifestly nondegenerate.

This correspondence suggests that the r^p-hierarchy near $\mathcal{H}^+$ should be of the following form (together with the standard r^p-hierarchy near $\mathcal{I}^+$) :

$$\begin{aligned} (1.14) \quad & \int_{C(\tau_2)} r^p (\partial_v \psi)^2 \, dv + \int_{\underline{C}(\tau_2)} (r - M)^{2-p} \frac{(\partial_u \psi)^2}{-\partial_u r} \, du \\ & \qquad \lesssim \int_{C(\tau_1)} r^p (\partial_v \psi)^2 \, dv + \int_{\underline{C}(\tau_1)} (r - M)^{2-p} \frac{(\partial_u \psi)^2}{-\partial_u r} \, du + \text{l.o.t.}, \end{aligned}$$

$$(1.15) \quad \int_{\tau_1}^{\tau_2} \int_{C(\tau)} r^{p-1} (\partial_v \psi)^2 \, dv d\tau \lesssim \int_{C(\tau_1)} r^p (\partial_v \psi)^2 \, dv + \text{l.o.t.},$$

$$(1.16) \quad \int_{\tau_1}^{\tau_2} \int_{\underline{C}(\tau)} (r - M)^{3-p} \frac{(\partial_u \psi)^2}{-\partial_u r} \, du d\tau \lesssim \int_{\underline{C}(\tau_1)} (r - M)^{2-p} \frac{(\partial_u \psi)^2}{-\partial_u r} \, du + \text{l.o.t.},$$

where (u, v) denote Eddington–Finkelstein double null coordinates on the domain of outer communication, τ is proper time along a timelike curve Γ with constant area-radius, $\tau_1 \leq \tau_2$, $p \in [0, 3)$ in

[1]The conformal energy $\mathcal{C}_{N_\tau^{\mathcal{H}}}$ in [AAG20, (5.12)] is different from the one in [AKU24, Section 1.2.1] up to a factor of r^2. It does not affect the conformality but we adopt the convention in [AKU24, Section 1.2.1] so that the role of Couch-Torrence inversion is clearer.

(1.14), $p \in [1,3)$ in (1.15) and (1.16), "l.o.t." denotes terms lower in the p-hierarchy, the foliations $C(\tau)$ and $\underline{C}(\tau)$ are defined pictorially in Figure 3 below. Using pigeonhole principle, (1.14), (1.16)

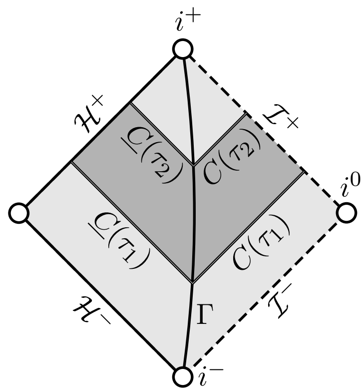

FIGURE 3. A Penrose diagram of extremal Reissner–Nordström depicting the foliations $C(\tau)$ and $\underline{C}(\tau)$ used in the estimates (1.14)–(1.16). The region of integration in (1.15) and (1.16) is shaded darker.

and (1.16) can be used to prove the energy decay estimate

$$\int_{C(\tau)} r^p (\partial_v \psi)^2 \, dv + \int_{\underline{C}(\tau)} (r-M)^{2-p} \frac{(\partial_u \psi)^2}{-\partial_u r} \, du \leq C_\star \tau^{-3+\delta+p},$$

for every $p \in [0, 3-\delta]$ and $\tau \geq \tau_0$, where $C_\star$ is a constant depending on δ and the data at $\tau = \tau_0$. This estimate can then be used to prove pointwise decay of ψ itself. Note that the non-degenerate energy when $p = 2$ on $\underline{C}(\tau)$ decays only like $\tau^{-1+\delta}$, which is in contrast with the subextremal case

Remark 1.1. The decay of $\tau^{-1+\delta}$ is sharp for generic data. See [AAG17, Theorem 3.2], where it is shown that the rate of decay of flux is not integrable in time when the Aretakis charge is nonzero. This rate turns out to persist in the nonlinear theory under a quantitative nonvanishing condition of asymptotic Aretakis charge (see [AKU24, Section 9.3]) and this slow decay is responsible for many of the technical difficulties.

2. Setup of the problem

We refer to the Penrose diagram (Figure 4) of the setup of the main results. An overview of what we need is described as follows :

- Gauge : An "initial data normalized gauge" $(\hat{u}, \hat{v})$ (see Section 2.1) and a renormalization $(u_{\tau_f}, v) = \Phi_{\tau_f}(\hat{u}, \hat{v})$ (see Section 2.2) are used. Note that $(\hat{u}, \hat{v})$ are regular across the event horizon and will be used to establish : existence of black hole regions, absence of trapped surfaces (weak cosmic censorship related statements in the main theorem) and Aretakis instability. On the other hand, (u_{τ_f}, v) only makes sense in the domain of outer communication and will be used to establish stability results in the region described in Figure 4.

  We also remark that the gauge (u_{τ_f}, v) is teleologically defined on the bootstrap region $\hat{\mathcal{D}}_{\tau_f}$. Therefore, an important final step when taking $\tau_f \to \infty$ is to prove that we can define a unique "final background solution" which we converge to and an "eschatological gauge" Φ_∞ (i.e., final teleological gauge) in which we converge to it. This gauge is defined on $\mathcal{D}_\infty$. (See Section 2.2) The regularity of the limiting gauge is actually related to the decay assumptions on initial data.

- Modulation : While our bootstrap argument is performed continuously in time, the choice of allowed modulation parameters α is only decided when τ_f is dyadic, i.e., a power of 2. This approach is motivated by the purely dyadic approach of Dafermos–Holzegel–Rodnianski–Taylor in [DHRT24] and turns out to be quite fortuitous compared to the continuous in time modulation theory employed in [DHRT21].
- Energy estimates : The energies are defined akin to the fixed background case above. However, in the near horizon region, the dynamical r and the background $\bar{r}$ need to be distinguished. See Remark 2.7.
- Estimates for the geometric quantities.
- Aretakis instability : Instead of establishing a conserved quantity (the Aretakis charge) as in the fixed ERN case, a notion called the asymptotic Aretakis charge is introduced in the dynamical setting. What is shown is that

$$\partial_v(Y\psi|_{\mathcal{H}^+}) = O(\epsilon^2\tau^{-2+\delta}),$$

where the rate is integrable in τ. Here, $Y := (\partial_u r)^{-1}\partial_u$ is gauge invariant null derivative transversal to $\mathcal{H}^+$.

2.1. **Seed data and characteristic initial data.** Bifurcate characteristic seed data for the spherically symmetric Einstein-Maxwell-neutral scalar field model on $C_{\text{out}} \cup \underline{C}_{\text{in}}$ is a quadruple $\mathcal{S} := (\mathring{\phi}, \Lambda = r_0, \varpi_0, e)$ consisting of

$$\mathring{\phi} := \phi|_{C_{\text{out}}\cup\underline{C}_{\text{in}}}, \quad r_0 := \text{ area-radius of } C_{\text{out}} \cup \underline{C}_{\text{in}},$$
$$\varpi_0 := \varpi|_{C_{\text{out}}\cup\underline{C}_{\text{in}}}, \quad e := \text{ the constant charge of the solution.}$$

On the other hand, a smooth *(bifurcate) characteristic initial data set* for the Einstein–Maxwell-scalar field system consists of a triple of smooth functions $\mathring{r}, \mathring{\Omega}^2, \mathring{\phi} : \mathcal{C} \to \mathbb{R}$ with $\mathring{r}$ and $\mathring{\Omega}^2$ positive, together with a real number e. The functions $\mathring{r}, \mathring{\Omega}^2$, and $\mathring{\phi}$ are assumed to satisfy (1.5) on $[U_0, U_1] \times \{V_0\}$ and (1.6) on $\{U_0\} \times [V_0, V_1]$.

Given $U_0 < U_1$ and $V_0 < V_1$, let

$$\mathcal{C}(U_0, U_1, V_0, V_1) := (\{U_0\} \times [V_0, V_1]) \cup ([U_0, U_1] \times \{V_0\}) \subset \mathbb{R}^{1+1},$$
$$\mathcal{R}(U_0, U_1, V_0, V_1) := [U_0, U_1] \times [V_0, V_1] \subset \mathbb{R}^{1+1}.$$

For any $f : \mathcal{C} \to \mathbb{R}$, we define $f_{\text{out}} := f|_{\{U_0\}\times[V_0,V_1]}$ and $f_{\text{in}} := f|_{[U_0,U_1]\times\{V_0\}}$

Proposition 2.1 (Generating characteristic initial data from seed data)**.** *Let $\mathcal{S} = (\mathring{\phi}, r_0, \varpi_0, e)$ be a suitable seed data set (close to $(0, 100M_0, M_0, e_0$ defined later) on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $U_1 - U_0 < r_0$. Then there exists a unique characteristic initial data set $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$ on $\mathcal{C}(U_0, U_1, V_0, V_1)$ such that the maximal development $(\mathcal{Q}_{\max}, r, \Omega^2, \phi, e)$ of $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$ has the following properties:*

(1) $r(U_0, V_0) = r_0$,
(2) $\varpi(U_0, V_0) = \varpi_0$,
(3) $\nu = -1$ on $[U_0, U_1] \times \{V_0\}$, and
(4) $\lambda = 1$ on $\{U_0\} \times [V_0, V_1]$.

We refer to characteristic data obtained from $\mathcal{S}$ in this manner as *gauge-normalized* characteristic data determined by $\mathcal{S}$.

Proof. Note that $\mathring{\phi}$ and e are already given in the seed data and hence we only need to generate $\mathring{r}$ and $\mathring{\Omega}^2$. For $(u, v) \in [U_0, U_1] \times [V_0, V_1]$, set

$$\mathring{r}_{\text{in}}(u) := r_0 - u + U_0, \quad \mathring{r}_{\text{out}}(v) := r_0 + v - V_0,$$

and then (1), (3) and (4) are satisfied. It is then motivated from Raychaudhuri's equations (1.5) and (1.6) that we choose

$$\mathring{\Omega}^2_{\text{in}}(u) := \mathring{\Omega}^2_0 \exp\left(-\int_{U_0}^{u} \mathring{r}_{\text{in}}(\partial_u \mathring{\phi}_{\text{in}})^2\, du'\right), \quad \mathring{\Omega}^2_{\text{out}}(v) := \mathring{\Omega}^2_0 \exp\left(\int_{V_0}^{v} \mathring{r}_{\text{out}}(\partial_v \mathring{\phi}_{\text{out}})^2\, dv'\right),$$

[2] where the freedom to choose the constant $\mathring{\Omega}^2_0$ allows us to make sure that (2) is satisfied. Namely, we choose $\mathring{\Omega}^2$ so that

$$\varpi_0 = \frac{r_0}{2}(1 - \frac{4}{\mathring{\Omega}^2_0}) + \frac{e^2}{2r_0}$$

thanks to (1.9) and the definition of renormalized Hawking mass. Note that when the seed data is close to ERN $(0, 100M_0, M_0, e_0)$ given below, $\mathring{\Omega}^2_0 > 0$.

Assembling these functions on $\mathcal{C}$ together will produce a characteristic data set $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$. □

Fix a mass parameter $M_0 > 0$ and let e_0 satisfy $|e_0| = M_0$. To initiate the study of stability, we consider perturbations of the bifurcate cone in ERN solution with parameters (M_0, e_0) with bifurcation sphere at $r = 100M_0$ satisfying $M_0 = |e_0|$. Namely, we consider the seed data $(0, 100M_0, M_0, e_0)$, which corresponds to ERN with parameters (M_0, e_0) and bifurcation sphere area-radius $100M_0$.

We set $U_* := 99.5M_0$ and denote

$$\hat{\mathcal{C}} := \mathcal{C}(0, U_*, 0, \infty) = \underline{C}_{\text{in}} \cup C_{\text{out}}$$

denote the bifurcate null hypersurface on which we pose our data. We denote the null coordinates on $\hat{\mathcal{C}}$ by $\hat{u}$ and $\hat{v}$. In view of Proposition 2.1, $(\hat{u}, \hat{v})$ are "initial data normalized" coordinates such that $\partial_{\hat{u}} r = -1$ on $\underline{C}_{\text{in}}$ and $\partial_{\hat{v}} r = 1$ on C_{out}. In particular, $\hat{u}$ will be regular across the event horizon.

Remark 2.2. It is usually not easy to establish stability in the "initial data normalized" gauge $\hat{u}$ and $\hat{v}$. They will be renormalized later in the proof. See Section 2.2. However, the "initial data normalized" gauge is indispensable when we consider the difference of two solutions, since the difference of data (which is coordinate dependent!) is originally given in these coordinates.

To characterize the perturbation, we consider the seed data norm

$$\mathfrak{D} \approx |r_0 - 100M_0| + |\varpi_0 - M_0| + |e - e_0| + \sup_{\underline{C}_{\text{in}}} \left(|\phi| + |\partial_{\hat{u}}\phi|\right) + \sup_{C_{\text{out}}} \left(|\psi| + |r^2 \partial_{\hat{v}} \psi|\right)$$

and define a master smallness parameter $\epsilon \geq \mathfrak{D}$. Precisely, given $S = (\mathring{\phi}, \Lambda_0, \varpi_0, e)$, the norm is given by

$$\begin{aligned}\mathfrak{D}[\mathcal{S}] := |\Lambda - 100M_0| + |\varpi_0 - M_0| + |e - e_0| + \sup_{\underline{C}_{\text{in}}} \left(|\mathring{\phi}_{\text{in}}| + |\partial_{\hat{u}} \mathring{\phi}_{\text{in}}|\right) \\ + \sup_{C_{\text{out}}} \left((1+\hat{v})|\mathring{\phi}_{\text{out}}| + (1+\hat{v})^2 |\partial_{\hat{v}} \mathring{\phi}_{\text{out}}| + (1+\hat{v})^2 |\partial_{\hat{v}}(\hat{v}\mathring{\phi}_{\text{out}})|\right).\end{aligned}$$

We claim that with seed data close to $(0, 100M_0, M_0, e_0)$, the evolution will behave nicely in view of Lemma 2.3. Moreover, Figure 4 illustrates the regions of interests. For the evolution (an ERN) from the seed data $(0, 100M_0, M_0, e_0)$ on $\hat{C}$, we have that the event horizon corresponds to

$$\hat{u}_{\mathcal{H}^+,0} := 99M_0 \tag{2.1}$$

in the $(\hat{u}, \hat{v})$ gauge.

[2]It seems that [AKU24] made some typos in their proof of this proposition.

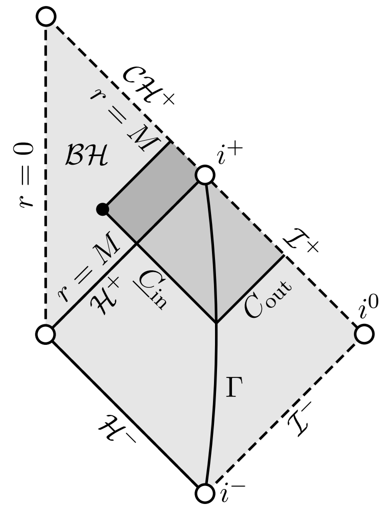

FIGURE 4. A Penrose diagram of (one period of) the maximally extended extremal Reissner–Nordström solution. The union of the two darker shaded regions is the domain of dependence of the bifurcate null hypersurface $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$ and represents the solution we are perturbing around in the main theorem. We prove stability of the medium gray colored region.

Lemma 2.3. *There exists an $\epsilon_{\mathrm{loc}} > 0$ depending only on M_0 such that if $\mathcal{S}$ is a seed data set for which*

$$\mathfrak{D}[\mathcal{S}] \leq 3\epsilon_{\mathrm{loc}},$$

then the maximal globally hyperbolic development $(\hat{\mathcal{Q}}_{\max}, r, \Omega^2, \phi, e)$ of $\mathcal{S}$ has $\partial_{\hat{u}} r < 0$ everywhere on $\hat{\mathcal{Q}}_{\max}$.

Definition 2.4. Let $M_0 > 0$. We define $\mathfrak{M}_0$ to be the set of all seed data with mass M_0. The *moduli space of seed data* centered on mass M_0 with *smallness parameter* ϵ is the set

$$\mathfrak{M}(\epsilon) := \bigcup_{\mathcal{S}_0 \in \mathfrak{M}_0 : \mathfrak{D}[\mathcal{S}_0] \leq \epsilon} \mathcal{L}(\mathcal{S}_0, \epsilon),$$

where

$$\mathcal{L}(\mathcal{S}_0, \epsilon) := \{\mathcal{S}_0(\alpha) : \alpha \in [-2\epsilon, 2\epsilon]\}, \quad \mathcal{S}_0(\alpha) := (\mathring{\phi}, \Lambda, M_0 + \alpha, e), \quad \mathcal{S}_0 := \mathcal{S}_0(0) \in \mathfrak{M}_0.$$

We endow $\mathfrak{M}(\epsilon)$ with the metric space topology associated to the norm $\mathfrak{D}$. Here, $\mathfrak{M}_0$ is a codimension-one affine subspace of the vector space of seed data and $\mathcal{L}(\mathcal{S}_0, \epsilon)$ is a line segment.

2.2. **The geometric setup for the stability statement.** In this subsection, we introduce a renormalization of the "initial data normalized" gauge. Consider ϵ_{loc} defined in Lemma 2.3. Let $\mathcal{S} \in \mathfrak{M}(\epsilon)$ with $0 < \epsilon \leq \epsilon_{\mathrm{loc}}$.

We define the set

$$\Gamma := \{r = \Lambda\}.$$

This is clearly a timelike curve near $\hat{\mathcal{C}}$ for ϵ sufficiently small and we will verify in the course of the proof of the main theorem that Γ is an inextendible timelike curve in $\hat{\mathcal{Q}}_{\max}$. Assuming for the moment that this is the case, we may parametrize Γ by its proper time τ, which we normalize to start at 1 at $\Gamma \cap \hat{\mathcal{C}}$. We write the components of Γ as $\Gamma(\tau) = (\Gamma^{\hat{u}}(\tau), \Gamma^{\hat{v}}(\tau))$ in the $(\hat{u}, \hat{v})$ coordinates on $\hat{\mathcal{Q}}_{\max}$.

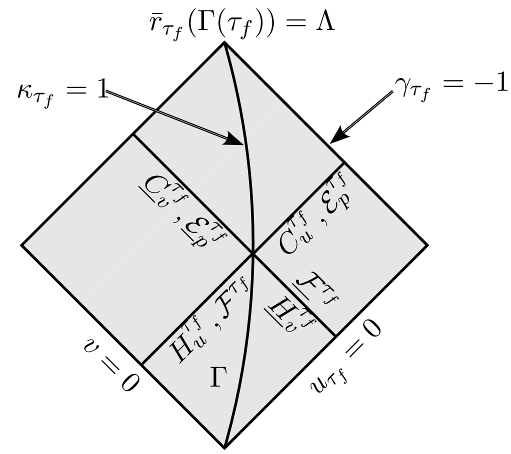

FIGURE 5. A Penrose diagram showing the gauge conditions, null hypersurfaces, and energies in our bootstrap domain $\mathcal{D}_{\tau_f}$. The function τ measures advanced time to the left of Γ and retarded time to the right of Γ.

We now wish to define a new gauge such that κ and γ are normalized along specific curves, respectively. See Figure 5. Note that we need to make a series of assumptions in order to define this gauge and these assumptions will be formalized as bootstrap assumptions later.

For $\tau_f \in [1, \infty)$ such that $[1, \tau_f]$ lies in the domain of definition Γ, we define

$$\hat{\mathcal{D}}_{\tau_f} := [0, \Gamma^{\hat{u}}(\tau_f)] \times [0, \Gamma^{\hat{v}}(\tau_f)].$$

Assuming that $\hat{\gamma} < 0$ on the final ingoing cone in $\hat{\mathcal{D}}_{\tau_f}$ (i.e., $\hat{\gamma} < 0$ when $\hat{v} = \Gamma^{\hat{v}}(\tau_f)$) and $\hat{\kappa} > 0$ on $\Gamma \cap \hat{\mathcal{D}}_{\tau_f}$. Then it implies that the two functions $\mathfrak{u}_{\tau_f} : [0, \Gamma^{\hat{u}}(\tau_f)] \to \mathbb{R}$ and $\mathfrak{v} : [0, \Gamma^{\hat{v}}(\tau_f)] \to \mathbb{R}$, defined via

$$\mathfrak{u}_{\tau_f}(\hat{u}) := -\int_0^{\hat{u}} \hat{\gamma}(\hat{u}', \Gamma^{\hat{v}}(\tau_f))\, d\hat{u}',$$
$$\mathfrak{v}(\hat{v}) := \int_0^{\hat{v}} \hat{\kappa}(\Gamma^{\hat{u}}((\Gamma^{\hat{v}})^{-1}(\hat{v}')), \hat{v}')\, d\hat{v}',$$

are strictly increasing on their domains, respectively. Therefore, they can be assembled into a map

$$\Phi_{\tau_f} : \hat{\mathcal{D}}_{\tau_f} \to \mathbb{R}^2, \qquad (\hat{u}, \hat{v}) \mapsto (\mathfrak{u}_{\tau_f}(\hat{u}), \mathfrak{v}(\hat{v})),$$

and it is a diffeomorphism onto its image. We denote the image of Φ_{τ_f} by $\mathcal{D}_{\tau_f}$, which comes equipped with the double null coordinates $(u_{\tau_f}, v) = \Phi_{\tau_f}(\hat{u}, \hat{v})$. Let $\hat{\Phi}_{\tau_f}$ denote the inverse of Φ_{τ_f}. In the (u_{τ_f}, v) coordinate system, the solution $(r, \hat{\Omega}^2, \phi, e)$ is given by $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$, where

$$r_{\tau_f} := r \circ \hat{\Phi}_{\tau_f}, \quad \phi_{\tau_f} := \phi \circ \hat{\Phi}_{\tau_f}, \quad \Omega^2_{\tau_f} := \frac{1}{\mathfrak{u}'_{\tau_f} \circ \mathfrak{u}^{-1}_{\tau_f}} \frac{1}{\mathfrak{v}' \circ \mathfrak{v}^{-1}} \hat{\Omega}^2 \circ \hat{\Phi}_{\tau_f}.$$

Indeed,

$$\Omega_{\tau_f}(u_{\tau_f}, v)^2\, du_{\tau_f}\, dv = (\Omega^2_{\tau_f} \circ \Phi_{\tau_f}) \mathfrak{u}'_{\tau_f}(\hat{u}) \mathfrak{v}'(\hat{v})\, d\hat{u}\, d\hat{v}.$$

In view of

$$\partial_{\hat{u}} u_{\tau_f} = -\hat{\gamma}(\hat{u}, \Gamma^{\hat{v}}(\tau_f)), \quad \partial_{\hat{v}} v = \hat{\kappa}(\Gamma^{\hat{u}}((\Gamma^{\hat{v}})^{-1}(\hat{v})), \hat{v}),$$

we know that

$$\gamma_{\tau_f}|_{\hat{v}=\Gamma^{\hat{v}}(\tau_f)} = \left.\frac{\partial_{u_{\tau_f}} r}{1 - \frac{2m}{r_{\tau_f}}}\right|_{\hat{v}=\Gamma^{\hat{v}}(\tau_f)} = (\partial_{\hat{u}} u_{\tau_f})^{-1} \hat{\gamma}(\hat{u}, \Gamma^{\hat{v}}(\tau_f)) = -1, \quad \kappa_{\tau_f}(\hat{u}, \hat{v})|_\Gamma = \left.\frac{\partial_v r}{1 - \frac{2m}{r_{\tau_f}}}\right|_{(\hat{u},\hat{v})=(\Gamma^{\hat{u}}((\Gamma^{\hat{v}})^{-1}(\hat{v})),\hat{v})} = 1.$$

This verifies the normalization shown in Figure 5.

Remark 2.5. The motivation of this normalization is natural due to (1.13) in the exact ERN setting. However, it is in contrast with the future-normalized coordinates in [LO19b, Section 5.1] in the subextremal case. In [LO19b], instead of normalizing along Γ, they normalize on the final outgoing cone as well. These future normalized coordinates are actually pretty natural to use for a stability result. However, closing the argument with such a gauge requires redshift estimates. Heuristically, in view of the asymptotic stability result that we need to prove, the final outgoing cone will finally converge to $\mathcal{H}^+$. In order to take advantage of the normalization, one would like to integrate geometric quantities along the final outgoing cone, on which the scalar field estimates are unfavorable in the extremal case. In [LO19b], this normalization works out thanks to the red-shift estimates.

On the other hand, in the companion paper concerning the interior of black holes, two additional coordinates are defined adapted to $\mathcal{CH}^+$ and $\mathcal{H}^+$, respectively. See [LO19a, Section 5.1], where one adapted to $\mathcal{H}^+$ is regular across $\mathcal{H}^+$. However, the regularity and its definition relies on the subextremality, which could not be generalized to ERN. Therefore, one expects that redshift effects play an important role when one wants to close the bootstrap argument.

We also remark that the gauge choice is also different from [DHRT21], where two gauges $\mathcal{I}^+$-gauge and $\mathcal{H}^+$-gauge are chosen.

In the (u_{τ_f}, v) coordinate system, we write the coordinates of Γ as $\Gamma(\tau) = (\Gamma^{u_{\tau_f}}(\tau), \Gamma^v(\tau))$. We will frequently omit the subscript τ_f on u_{τ_f} and $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$ when it is clear that τ_f has been fixed.

Some basic properties are then derived in [AKU24, Section 5.2.3] based on the bootstrap assumptions. In particular, we wish to ensure that

$$\dot{\Gamma}^u(\tau) \sim \dot{\Gamma}^v(\tau) \sim 1$$

for $\tau \in [1, \tau_f]$ when designing the bootstrap assumptions. Here, τ is used for the proper time parametrization of Γ, i.e.,

$$g(\dot{\Gamma}, \dot{\Gamma}) = -1, \tag{2.2}$$

where $\dot{}$ is used to denote $\frac{d}{d\tau}$.

(2.3) suggests that to define a continuous function $\tau = \tau(u, v)$ on $\mathcal{D}_{\tau_f}$ (this is a slight abuse of notation) implicitly by :

$$\tau(u, v) := \begin{cases} \tau : \Gamma^u(\tau) = u & \text{if } r(u, v) \geq \Lambda \\ \tau : \Gamma^v(\tau) = v & \text{if } r(u, v) < \Lambda \end{cases}.$$

Namely, this is obtained by extending the proper time τ to a function on $\mathcal{D}_{\tau_f}$ by setting it equal to proper time along Γ and then declaring it to be constant along ingoing cones to the left of Γ and constant along outgoing cones to the right of Γ.

Thanks to (2.3), the function τ measures (approximately) Bondi time in the region near null infinity and (approximately) Eddington–Finkelstein time near the event horizon. More precisely, (2.3), together with the fundamental theorem of calculus and the change of variables formula, implies that

$$\tau(u_1, v) - \tau(u_2, v) \sim u_1 - u_2$$

for $(u_1, v), (u_2, v) \in \mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$ and

$$\tau(u, v_1) - \tau(u, v_2) \sim v_1 - v_2$$

for $(u,v_1),(u,v_2)\in\mathcal{D}_{\tau_f}\cap\{r\le\Lambda\}$. Moreover, for any $\eta>1$ and $1\le\tau_1\le\tau_2\le\tau_f$, it holds that

$$\int_{\underline{H}_v\cap\{\tau_1\le\tau\le\tau_2\}}\tau^{-\eta}\,du\lesssim_\eta\tau_1^{-\eta+1},$$
$$\int_{H_u\cap\{\tau_1\le\tau\le\tau_2\}}\tau^{-\eta}\,dv\lesssim_\eta\tau_1^{-\eta+1}.$$

Now we define the region $\mathcal{D}_\infty$ (the region as $\tau_f\to\infty$) equipped with the "eschatological gauge".

For the seed data sets we will ultimately consider, Γ exists and remains timelike for all $\tau\in[1,\infty)$. We then define a number

$$\hat{u}_{\mathcal{H}^+}:=\lim_{\tau\to\infty}\Gamma^{\hat{u}}(\tau).$$

Since $\tau\mapsto\Gamma^{\hat{u}}(\tau)$ is monotone increasing, the existence of this limit is automatic and we will show the strict inequality

$$\hat{u}_{\mathcal{H}^+}<U_*,$$

see already Lemma 3.7. We then set

$$\hat{\mathcal{D}}_\infty:=[0,\hat{u}_{\mathcal{H}^+})\times[0,\infty).$$

We will also show that there exists a surjective, strictly increasing C^1 function $u_\infty:[0,\hat{u}_{\mathcal{H}^+})\to[0,\infty)$, such that, if we use (u_∞,v) as coordinates on $\hat{\mathcal{D}}_\infty$, then $\partial_{u_\infty}r\to-1$ at null infinity $\mathcal{I}^+$. We denote $\hat{\mathcal{D}}_\infty$ by $\mathcal{D}_\infty$ under this change of coordinates.

2.3. **Anchored extremal Reissner–Nordström solutions and definitions of the energies.**

Let (r,Ω^2,ϕ,e) be a spherically symmetric solution of the Einstein–Maxwell-scalar field system defined on a coordinate rectangle $\mathcal{D}_{\tau_f}$ with gauge conditions as explained in Section 2.2.

2.3.1. *The case $\tau_f<\infty$.* First, assume $\tau_f<\infty$. We define the *τ_f-anchored background solution* to be the extremal Reissner–Nordström metric $(\bar{r}_{\tau_f},\bar{\Omega}^2_{\tau_f})$ with parameters $M=|e|$ in Eddington–Finkelstein double null form which is uniquely determined by

$$\bar{r}_{\tau_f}(\Gamma(\tau_f))=\Lambda.$$

(This is done by simply changing the origin of the double-null coordinates, see [AKU24, Lemma 2.2].)

Given the anchored background solution we now adopt the following notation:

- Barred quantities such as $\bar{\lambda}_{\tau_f}$, $\bar{\varpi}_{\tau_f}=M$, or $\bar{\kappa}_{\tau_f}=1$ correspond to those of $(\bar{r}_{\tau_f},\bar{\Omega}^2_{\tau_f})$.
- Differences are denoted with a tilde, such as $\tilde{r}_{\tau_f}:=r_{\tau_f}-\bar{r}_{\tau_f}$, $\tilde{\varpi}_{\tau_f}:=\varpi_{\tau_f}-M$, or $\tilde{\gamma}_{\tau_f}:=\gamma_{\tau_f}+1$.

2.3.2. *The case $\tau_f=\infty$.* In the proof of the main theorem, we will send $\tau_f\to\infty$ and thus need to extend this definition to the case when $\tau_f=\infty$. Instead of trying to anchor directly "at $\tau_f=\infty$," it is much easier to simply anchor the background solution at $\Gamma\cap\hat{\mathcal{C}}=\{(0,0)\}$ and then show that it is compatible with directly anchoring at $\tau_f=\infty$.

We define the *∞-anchored background solution* to be the unique extremal Reissner–Nordström metric $(\bar{r}_\infty,\bar{\Omega}^2_\infty)$ with parameters $M=|e|$ in Eddington–Finkelstein double null form with the property that

$$\bar{r}_\infty(0,0)=\bar{r}_\star:=\lim_{\tau_f\to\infty}\bar{r}_{\tau_f}(0,0),$$

see already Figure 6. It turns out that this limit actually exists. We will also show that this background solution is anchored "at $\tau_f=\infty$" in the sense that

$$\lim_{\tau_f\to\infty}\bar{r}_\infty(\Gamma(\tau_f))=\Lambda.$$

2.3.3. *Energy fluxes in the dynamical setting.* We may now define the fundamental weighted energy norms for the scalar field. Some of the norms will depend explicitly on the background solution $\bar{r}_{\tau_f}$ in a nontrivial manner.

Definition 2.6 (A rough statement of [AKU24, Definition 3.5])**.** Let $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$ be defined on $\mathcal{D}_{\tau_f}$ with teleologically normalized coordinates (u_{τ_f}, v), where $\tau_f \in [1, \infty]$. Let $\bar{r}_{\tau_f}$ be the associated τ_f-anchored background solution. Let

$$\psi_{\tau_f} := r_{\tau_f}\phi_{\tau_f}$$

denote the *radiation field* of ϕ_{τ_f}. For $\tau, \tau' \in [1, \tau_f]$, $p \in [0, 3)$, and $(u, v) \in \mathcal{D}_{\tau_f}$, we define:

(1) The $(\bar{r} - M)^{2-p}$-weighted flux to the horizon:

$$\underline{\mathcal{E}}^{\tau_f}_p(\tau) := \int_{\underline{C}^{\tau_f}(\tau)} \left((\bar{r}_{\tau_f} - M)^{2-p} \frac{(\partial_{u_{\tau_f}}\psi_{\tau_f})^2}{-\bar{\nu}_{\tau_f}} + \cdots \right) du_{\tau_f}.$$

(2) The r^p-weighted flux to null infinity:

$$\mathcal{E}^{\tau_f}_p(\tau) := \int_{C^{\tau_f}(\tau)} \left((r^p_{\tau_f}(\partial_v\psi_{\tau_f})^2 + \cdots \right) dv.$$

Remark 2.7. It is mentioned in [AKU24, Remark 1.5] that the usage of $\bar{r}_{\tau_f} - M$ is crucial. Here is an attempt to conclude why this is the case and what are the bootstrap assumptions we need :

- $\bar{r} - M \geq 0$ has a sign, which is used, for instance, in [AKU24, Proof of Lemma 6.6]. This condition holds according to how we anchor our background ERN and [AKU24, Lemma 2.2].
- $\bar{\nu} = \partial_u \bar{r} = (1 - \frac{M}{\bar{r}})^2$ has explicit cancellation with $(\bar{r} - M)$-weight in the energy norms.
- The previous point also suggests the need of an estimate of $|\frac{\nu}{\bar{\nu}} - 1|$ in the bootstrap assumptions since to pass a geometric computation in dynamical variables to the background variable in the energy norms, one needs

$$\frac{(\partial_u\psi)^2}{-\nu} = (\frac{\bar{\nu}}{\nu} - 1)\frac{(\partial_u\psi)^2}{-\bar{\nu}} + \frac{(\partial_u\psi)^2}{-\bar{\nu}}.$$

  For the same reason, it is natural to require an estimate of $|r - \bar{r}|$.
- Besides, the requirement of a quantitative decay of $|\varpi - M|$ in the bootstrap assumptions comes from the nested bootstrap parameter set $\mathfrak{A}_i$'s and will be clear in the following.

2.4. **The bootstrap and modulation parameter sets.** We define in this section two sets of parameters: a bootstrap set $\mathfrak{B}$ containing the τ_f's for which we assume the solution exists on $\mathcal{D}_{\tau_f}$ and satisfies certain properties, and a sequence of compact intervals

$$\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots$$

of α parameters which are used in the modulation argument to hit extremality.

We first set

$$\mathfrak{A}_0 := [|e| - |e_0| - \epsilon^{3/2}, |e| - |e_0| + \epsilon^{3/2}],$$

but the sets $\mathfrak{A}_i$ for $i \geq 1$ will only be properly defined in the course of the proof of main theorem.

It is convenient to define the function $I(\tau_f) := \lfloor \log_2 \tau_f \rfloor$, i.e., the largest integer such that $2^{I(\tau_f)} \leq \tau_f$.

Definition 2.8 (A rough version of [AKU24, Definition 4.1])**.** Let $A \geq 1$, $0 < \epsilon \leq \epsilon_{\mathrm{loc}}$, and $\mathcal{S}_0 \in \mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0] \leq \epsilon$. Using A to denote the bootstrap constant and ϵ to denote the smallness inherited from initial data, then $\mathfrak{B}(\mathcal{S}_0, \epsilon, A)$ denotes the set of bootstrap times $\tau_f \in [1, \infty)$ such that :

- (Existence of nested intervals for the parameter α.) For every $i \in \{0, 1, \dots, I(\tau_f)\}$, there exist numbers $\alpha_i^\pm \in [|e| - |e_0| - \epsilon^{3/2}, |e| - |e_0| + \epsilon^{3/2}]$, which may depend on $\mathcal{S}_0$ and ϵ, but are independent of A and τ_f, with $\alpha_i^- < \alpha_i^+$ and $\alpha_0^\pm = |e| - |e_0| \pm \epsilon^{3/2}$, such that the nesting condition $\mathfrak{A}_{i+1} \subset \mathfrak{A}_i$ holds, where $\mathfrak{A}_i := [\alpha_i^-, \alpha_i^+]$.
- Given $\alpha \in \mathfrak{A}_{I(\tau_f)}$, let $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ denote the maximal development of the modulated seed data $\mathcal{S}_0(\alpha)$ in the initial data gauge $(\hat{u}, \hat{v})$ determined by Proposition 2.1 and Lemma 2.3.
- (Existence of the timelike curve $r = \Lambda$.) For every $\alpha \in \mathfrak{A}_{I(\tau_f)}$, there exists a timelike curve $\Gamma : [1, \tau_f] \to \hat{\mathcal{Q}}_{\max}$, which is the unique smooth solution of the ODE

$$\frac{d}{d\tau}(\Gamma^{\hat{u}}, \Gamma^{\hat{v}}) = \left(\frac{\sqrt{1-\mu}}{-2\hat{\nu}}, \frac{\sqrt{1-\mu}}{2\hat{\lambda}} \right)\Big|_{\Gamma(\tau)},$$

with initial condition $\Gamma(1) = (0, 0)$. (As a consequence of the global hyperbolicity, $J^+(\Gamma(1)) \cap J^-(\Gamma(\tau_f)) = \hat{D}_{\tau_f} := [0, \Gamma^{\hat{u}}(\tau_f)] \times [0, \Gamma^{\hat{v}}(\tau_f)] \subset \hat{\mathcal{Q}}_{\max}$.)
- (Saturated estimate on dyadic scale for renormalized mass.) For every $i \in \{0, 1, \dots, I(\tau_f)\}$, the map

$$\Pi_i := \alpha \mapsto \varpi(\Gamma(L_i)) - M, \quad \mathfrak{A}_i \to [-\epsilon^{-3/2} L_i^{-3+\delta}, \epsilon^{3/2} L_i^{-3+\delta}] \subset \mathbb{R}, \quad L_i := 2^i$$

is defined on $\mathfrak{A}_i$ and is surjective onto $[-\epsilon^{-3/2} L_i^{-3+\delta}, \epsilon^{3/2} L_i^{-3+\delta}]$.
- (Sign condition.) For every $\alpha \in \mathfrak{A}_{I(\tau_f)}$, $\hat{\gamma}$ is strictly negative on $[0, \Gamma^{\hat{u}}(\tau_f)] \times \{\Gamma^{\hat{v}}(\tau_f)\}$ and $\hat{\kappa}$ is strictly positive on Γ. Therefore, the teleologically normalized coordinates (u_{τ_f}, v) are defined on $\hat{\mathcal{D}}_{\tau_f}$.
- (Bootstrap for the geometry.) A quantitative decay statement of $|\frac{\nu_{\tau_f}}{\bar{\nu}_{\tau_f}} - 1|$, $|r_{\tau_f} - \bar{r}_{\tau_f}|$ and $|\varpi_{\tau_f} - M|$ as mentioned in Remark 2.7 as a bootstrap assumption that needs an improvement.
- (Bootstrap for the scalar field.) A quantitative decay statement of p-weighted energy flux $\underline{\mathcal{E}}_p^{\tau_f}(\tau)$, $\mathcal{E}_p^{\tau_f}(\tau)$ as another bootstrap assumption that needs an improvement.

Remark 2.9. In the existence of the timelike curve $r = \Lambda$, we refer to the proof of Lemma 2.12 for the derivation of the ODE.

Remark 2.10. I feel that the saturated estimate on dyadic scale for renormalized mass is somehow similar to the trapping assumption in [Tan24]. However, with this nested set consideration, [AKU24] is only able to show that there exists a nonempty set of α and name this the so-called "codimension-one" result. See [AKU24, Remark 8.7].

Remark 2.11 (Proof of nonlinear stability). Besides improving the two sets of estimates, we comment on the continuity argument on $\mathfrak{B}$.

- By choosing $|\tau_f - 1|$ small, $\mathfrak{B} \neq \varnothing$.
- The openness of $\mathfrak{B}$ consists of a local-wellposedness theory of characteristic initial data problem and the simple property that $I(\tau_f + \eta) = I(\tau_f)$ for η sufficiently small. See [AKU24, Section 8.1.1].
- The closedness of $\mathfrak{B}$ is usually trivial (in view of the quantitative estimates part). It is slightly nontrivial for the construction of nested intervals $\mathfrak{A}_i$'s. Let $\tau_f^n \in \mathfrak{B}(\mathcal{S}_0, \epsilon, A_0)$ be a strictly increasing sequence of times with finite limit τ_f^∞ as $n \to \infty$. We aim to show that $\tau_f^\infty \in \mathfrak{B}(\mathcal{S}_0, \epsilon, A_0)$.

  When τ_f^∞ is not dyadic, it is trivial based on geometric estimates. When τ_f^∞ is dyadic, $I := I(\tau_f^\infty) > I(\tau_f^n)$ for any finite n. This requires us to construct $\mathfrak{A}_I$. Here, we need to use intermediate value theorem and hence the construction is implicit. See [AKU24, Lemma 8.4].

Lemma 2.12. *For any $A \geq 1$, ϵ sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\dot{\Gamma}^u(\tau) \sim \dot{\Gamma}^v(\tau) \sim 1 \tag{2.3}$$

for $\tau \in [1, \tau_f]$.

Proof. Since r is constant along Γ, we have $\nu\dot{\Gamma}^u + \lambda\dot{\Gamma}^v = 0$ on Γ. The proper time condition (2.2) can be written as

$$\frac{4\nu\lambda}{1-\mu}\dot{\Gamma}^u\dot{\Gamma}^v = -1,$$

which implies that

$$\dot{\Gamma}^u = \frac{\sqrt{1-\mu}}{-2\nu}, \quad \dot{\Gamma}^v = \frac{\sqrt{1-\mu}}{2\lambda} \text{ on } \Gamma.$$

By the gauge condition $\lambda = \kappa(1-\mu)|_\Gamma = 1-\mu|_\Gamma \sim 1$. Moreover, we have (3.6) and the result follows. $\square$

3. Main theorem and its proof

3.1. **Detailed statements of the main theorems.**

We can now state our main theorems using the notation and definitions.

3.1.1. *Nonlinear stability.*

Theorem 3.1 (Stability of extremal Reissner–Nordström in spherical symmetry)**.** *Let $M_0 > 0$, $e_0 \in \mathbb{R}$ with $|e_0| = M_0$, and let δ be an arbitrary parameter satisfying*

$$0 < \delta < \frac{1}{100}.$$

There exists a number $\epsilon_{\mathrm{stab}}(M_0, \delta) > 0$, a set $\mathfrak{M}_{\mathrm{stab}} \subset \mathfrak{M}$, and a constant $C(M_0, \delta)$ (which is implicit in the notation $\lesssim$ below) with the following properties:

(1) $\mathfrak{M}_{\mathrm{stab}}$ is "codimension-one" inside of $\mathfrak{M}(\epsilon)$: For every $0 < \epsilon \leq \epsilon_{\mathrm{stab}}$ and $\mathcal{S}_0 \in \mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0] \leq \epsilon$, it holds that

$$\mathfrak{M}_{\mathrm{stab}} \cap \mathcal{L}(\mathcal{S}_0, \epsilon) \neq \emptyset. \tag{3.1}$$

(2) Existence of a black hole region: Let $(\hat{\mathcal{Q}}_{\max}, r, \Omega^2, \phi, e)$ be the maximal development of a seed data set in the intersection (3.1). Then $\hat{\mathcal{Q}}_{\max} = [0, U_*) \times [0, \infty)$. There exists a $\hat{u}_{\mathcal{H}^+} \in (0, U_*)$ such that $r(\hat{u}, \hat{v}) \to \infty$ as $\hat{v} \to \infty$ for every for every $\hat{u} \in [0, \hat{u}_{\mathcal{H}^+})$ and $r(\hat{u}_{\mathcal{H}^+}, \hat{v}) \to |e|$ as $\hat{v} \to \infty$. Therefore, $[0, \hat{u}_{\mathcal{H}^+}) \times \{\hat{v} = \infty\}$ may be regarded as future null infinity $\mathcal{I}^+$, there exists a nonempty black hole region

$$\mathcal{BH} := \hat{\mathcal{Q}}_{\max} \setminus J^-(\mathcal{I}^+) = [\hat{u}_{\mathcal{H}^+}, U_*] \times [0, \infty),$$

and

$$\mathcal{H}^+ := \partial J^-(\mathcal{I}^+) = \{\hat{u}_{\mathcal{H}^+}\} \times [0, \infty)$$

is the event horizon. Moreover, future null infinity is complete in the sense of Christodoulou. There exist C^1 double null coordinates (u_∞, v) on the domain of outer communication $[0, \hat{u}_{\mathcal{H}^+}) \times [0, \infty)$ such that u_∞ is Bondi normalized, i.e., the event horizon $\mathcal{H}^+$ can be formally regarded as $\{u_\infty = \infty\}$ and $\partial_{u_\infty} r \to -1$ along any outgoing cone in the domain of outer communication.

(3) Orbital stability: There exists an ∞-anchored extremal Reissner–Nordström solution $\bar{r}_\infty$ in the (u_∞, v) coordinates whose parameters satisfy

$$|M - M_0| + |e - e_0| \lesssim \epsilon.$$

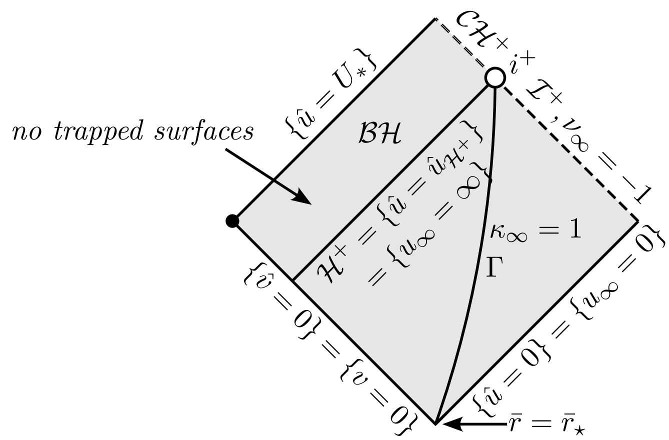

FIGURE 6. A Penrose diagram depicting the maximal development of seed data lying in $\mathfrak{M}_{\mathrm{stab}}$.

Relative to this background solution, the $p = 3-\delta$ energy of the scalar field is bounded by its initial value,

$$\sup_{\tau\in[1,\infty)} \left(\mathcal{E}^\infty_{3-\delta}(\tau) + \underline{\mathcal{E}}^\infty_{3-\delta}(\tau)\right) \lesssim \mathcal{E}^\infty_{3-\delta}(1) + \underline{\mathcal{E}}^\infty_{3-\delta}(1), \tag{3.2}$$

and the scalar field is pointwise bounded in C^1 in the domain of outer communication in terms of its initial values,

$$\begin{aligned} &\sup_{J^-(\mathcal{I}^+)} \left(|r\phi| + |r^2\partial_{\hat{v}}\psi| + |r^2\partial_{\hat{v}}\phi| + \left|\frac{\partial_{\hat{u}}\psi}{-\hat{\nu}}\right| + \left|r\frac{\partial_{\hat{u}}\phi}{-\hat{\nu}}\right|\right) \\ &\qquad\qquad \lesssim \sup_{J^-(\mathcal{I}^+)\cap\hat{\mathcal{C}}} \left(|r\phi| + |r^2\partial_{\hat{v}}\psi| + |r^2\partial_{\hat{v}}\phi| + \left|\frac{\partial_{\hat{u}}\psi}{-\hat{\nu}}\right| + \left|r\frac{\partial_{\hat{u}}\phi}{-\hat{\nu}}\right|\right). \end{aligned} \tag{3.3}$$

The right-hand side of (3.2) is $\lesssim \epsilon^2$ and the right-hand side of (3.3) is $\lesssim \epsilon$.

(4) *Asymptotic stability: The geometry decays towards the ∞-anchored extremal Reissner–Nordström solution in the following sense:*

$$|\gamma_\infty + 1| \lesssim \epsilon^2 r^{-1}\tau^{-3+\delta}$$

holds on $\mathcal{D}_\infty \cap \{r \geq \Lambda\}$,

$$\left|\frac{\nu_\infty}{\bar{\nu}_\infty} - 1\right| \lesssim \epsilon^2\tau^{-1+\delta}$$

holds on $\mathcal{D}_\infty$, and

$$|r - \bar{r}_\infty| \lesssim \epsilon^2\tau^{-2+\delta}, \quad |\lambda_\infty - \bar{\lambda}_\infty| \lesssim \epsilon^2\tau^{-2+\delta}, \quad |\kappa_\infty - 1| \lesssim \epsilon^2\tau^{-1+\delta}, \quad |\varpi - M| \lesssim \epsilon^2\tau^{-3+\delta}$$

hold on $\mathcal{D}_\infty$, up to and including the event horizon $\mathcal{H}^+$. The scalar field decays to zero in energy norm,

$$\underline{\mathcal{E}}^\infty_p(\tau) + \mathcal{E}^\infty_p(\tau) \lesssim \epsilon^2\tau^{-3+\delta+p}$$

for every $\tau \in [1,\infty)$ and $p \in [0, 3-\delta]$ and its amplitude decays to zero pointwise,

$$|(\bar{r}_\infty - M)^{1/2}\phi| \lesssim \epsilon\tau^{-3/2+\delta/2}, \quad |\psi| \lesssim \epsilon\tau^{-1+\delta/2}$$

on $\mathcal{D}_\infty$, up to and including the event horizon $\mathcal{H}^+$.

(5) *Absence of trapped surfaces: On the spacetime $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2)$, it holds that $\hat{\nu} < 0$, i.e., there exist no antitrapped spheres of symmetry. Moreover, we have the following dichotomy:*

(a) $\hat{\lambda} \geq 0$ *on* $\hat{\mathcal{Q}}_{\max}$*, i.e., there exist no (strictly) trapped symmetry spheres.*

(b) *If* $\lambda(\hat{u}_0, v_0) = 0$ *for some* $(\hat{u}_0, \hat{v}_0) \in \hat{\mathcal{Q}}_{\max}$*, then* $\hat{u}_0 = \hat{u}_{\mathcal{H}^+}$ *and* $\phi(\hat{u}_{\mathcal{H}^+}, \hat{v}) = 0$ *for all* $v \in [v_0, \infty)$.

3.1.2. *The Aretakis instability.* Let

$$Y := \hat{\nu}^{-1}\partial_{\hat{u}}$$

denote the gauge-invariant null derivative which is transverse to the event horizon $\mathcal{H}^+$, analogous to ∂_r in ingoing Eddington–Finkelstein coordinates (v, r) in Reissner–Nordström.

Theorem 3.2 (The Aretakis instability for dynamical extremal horizons)**.** *Let* $\mathfrak{M}_{\mathrm{stab}}$ *denote the subset of the moduli space* $\mathfrak{M}$ *given by Theorem 3.1 consisting of seed data asymptotically converging to extremal Reissner–Nordström in evolution. Then the following holds:*

(1) *For any solution* $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ *arising from* $\mathcal{S} \in \mathfrak{M}_{\mathrm{stab}}$*, the "asymptotic Aretakis charge"*

$$H_0[\phi] := \lim_{\hat{v}\to\infty} Y\psi|_{\mathcal{H}^+}$$

exists and it holds that

$$|Y\psi|_{\mathcal{H}^+}(\hat{v}) - H_0[\phi]| \lesssim \epsilon^3(1+\hat{v})^{-1+\delta},$$
$$\big|R_{YY}|_{\mathcal{H}^+}(\hat{v}) - 2M^{-2}\big(H_0[\phi]\big)^2\big| \lesssim \epsilon^2(1+\hat{v})^{-1+\delta/2},$$

where $\epsilon \geq \mathfrak{D}[\mathcal{S}]$.

(2) *The set*

$$\mathfrak{M}^{\neq 0}_{\mathrm{stab}} := \{\mathcal{S} \in \mathfrak{M}_{\mathrm{stab}} : H_0[\phi] \neq 0\}$$

has nonempty interior as a subset of $\mathfrak{M}_{\mathrm{stab}}$.

(3) *For any solution arising from data lying in* $\mathfrak{M}^{\neq 0}_{\mathrm{stab}}$*, it holds that*

$$\big|Y^2(r\phi)|_{\mathcal{H}^+}(\hat{v})\big| \gtrsim |H_0[\phi]|\hat{v},$$
$$\big|\nabla_Y R_{YY}|_{\mathcal{H}^+}(\hat{v})\big| \gtrsim \big(H_0[\phi]\big)^2\hat{v}$$

for $\hat{v} \gtrsim 1 + |\epsilon H_0[\phi]^{-1}|^{1/(1-\delta)}$.

3.2. **Completeness of null infinity.** We now prove the existence of a black hole region and a regular event horizon in the maximal development $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ of seed data lying in $\mathfrak{M}_{\mathrm{stab}}$.

Proposition 3.3. *The maximal development of any seed data lying in* $\mathfrak{M}_{\mathrm{stab}}$ *has* $\hat{\mathcal{Q}}_{\max} = [0, U_*] \times [0, \infty)$ *and there exists a* $\hat{u}_{\mathcal{H}^+} \in (0, U_*)$ *satisfying*

$$|\hat{u}_{\mathcal{H}^+} - \hat{u}_{\mathcal{H}^+,0}| \lesssim \epsilon, \tag{3.4}$$

where $\hat{u}_{\mathcal{H}^+,0}$ *was defined in (2.1), such that* $[0, \hat{u}_{\mathcal{H}^+}] \times [0, \infty) \subset \hat{\mathcal{Q}}_{\max}$,

$$\lim_{\hat{v}\to\infty} r(\hat{u}, \hat{v}) = \infty$$

for every $\hat{u} \in [0, \hat{u}_{\mathcal{H}^+})$*, and*

$$\lim_{\hat{v}\to\infty} r(\hat{u}_{\mathcal{H}^+}, \hat{v}) = \lim_{\hat{v}\to\infty} \varpi(\hat{u}_{\mathcal{H}^+}, \hat{v}) = M = |e|.$$

Therefore, $[0, \hat{u}_{\mathcal{H}^+}) \times \{\hat{v} = \infty\}$ *may be regarded as future null infinity* $\mathcal{I}^+$ *and*

$$\mathcal{H}^+ := J^-(\mathcal{I}^+) = \{\hat{u}_{\mathcal{H}^+}\} \times [0, \infty)$$

is the event horizon. The black hole region is

$$\mathcal{BH} := \hat{\mathcal{Q}}_{\max} \setminus J^-(\mathcal{I}^+) = [\hat{u}_{\mathcal{H}^+}, U_*] \times [0, \infty) \neq \emptyset$$

and future null infinity is complete in the sense of Christodoulou.

Remark 3.4. Note that this verifies that the limit along event horizon is extremal in the sense that

$$\lim_{\hat{v}\to\infty} \varpi(\hat{u}_{\mathcal{H}^+}, \hat{v}) = |e|.$$

This is compatible with the definition of subextremality of [LO19a].

Proof. By the geometric estimates, r is bounded from below on $\hat{\mathcal{D}}_\infty$ and $\hat{\lambda} > 0$. It follows from the extension principle [Daf05b] that $\overline{\hat{\mathcal{D}}_\infty} \subset \hat{\mathcal{Q}}_{\max}$, where the closure is taken in the $(\hat{u}, \hat{v})$-plane. By Lemma 3.7, $\overline{\hat{\mathcal{D}}_\infty} \setminus \hat{\mathcal{D}}_\infty = \{\hat{u}_{\mathcal{H}^+}\} \times [0, \infty)$.

The limit is proved by combining a decay estimates of $|r - \bar{r}_\infty \circ \Phi_\infty|$ and the geometry of ERN. For the estimate (3.4), see Lemma 3.5.

To show that $\hat{\mathcal{Q}}_{\max}$ contains the rectangle $(\hat{u}_{\mathcal{H}^+}, U_*] \times [0, \infty)$, we use the logically independent fact that $\hat{\lambda} \geq 0$ everywhere on $\hat{\mathcal{Q}}_{\max}$, which will be shown in [AKU24, Section 8.3.5] below. Since r is bounded below on $[0, U_*] \times \{0\}$, is it bounded below on $\hat{\mathcal{Q}}_{\max}$. The extension principle [Daf05b] therefore implies that $\hat{\mathcal{Q}}_{\max} = [0, U_*] \times [0, \infty)$.

Finally, since r is bounded on $\mathcal{H}^+$, completeness of $\mathcal{I}^+$ follows from the work of Dafermos [Daf05c]. □

3.3. **Absence of trapped surfaces.** We first recall the notions (see [Daf05c]) of *apparent horizon*

$$\mathcal{A} := \{(\hat{u}, \hat{v}) \in \hat{\mathcal{Q}}_{\max} : \lambda(\hat{u}, \hat{v}) = 0\},$$

the *regular region* or *non-trapped region*

$$\mathcal{R} := \{(\hat{u}, \hat{v}) \in \hat{\mathcal{Q}}_{\max} : \lambda(\hat{u}, \hat{v}) > 0\}$$

and *outermost apparent horizon*

$$\mathcal{A}' := \{(\hat{u}, \hat{v}) \in \mathcal{A} : (\hat{u}', \hat{v}) \in \mathcal{R} \text{ for every } \hat{u}' < \hat{u}\} = \{(\hat{u}, \hat{v}) \in \hat{\mathcal{Q}}_{\max} : \lambda(\hat{u}, \hat{v}) = 0 \text{ and } \lambda(\hat{u}', \hat{v}) > 0 \text{ for every } \hat{u}' < \hat{u}\}.$$

We observe from (1.10), (1.11) and $1 - \mu = -\frac{4\nu\lambda}{\Omega^2}$ that if $(\hat{u}, \hat{v}) \in \mathcal{R} \cup \mathcal{A}$, then we have the monotonicities

$$\partial_u \varpi(\hat{u}, \hat{v}) \leq 0 \quad \text{and} \quad \partial_v \varpi(\hat{u}, \hat{v}) \geq 0. \tag{3.5}$$

Proof that $\mathcal{A}' \subset \mathcal{H}^+$: First, it is easy to establish via contradiction that $\mathcal{H}^+ \subset \mathcal{R} \cup \mathcal{A}$.

Let $(\hat{u}_0, \hat{v}_0) \in \mathcal{A}'$.[3] By the monotonicities (3.5) and the outermost property of $(\hat{u}_0, \hat{v}_0)$, it follows that $M \geq \varpi(\hat{u}_0, \hat{v}_0)$. (This derivation is akin to [Daf05c, Proof of Lemma 2], one take some point on $\mathcal{I}^+$ with renormalized mass M'. Then $M' \geq M$ and one can integrate (3.5) to establish $M' \geq \varpi(\hat{u}_0, \hat{v}_0)$. Since any $M' \geq M$ works so this proves the inequality.)

We also have $1-\mu(\hat{u}_0, \hat{v}_0) = 0$ thanks to $1-\mu = -\frac{4\nu\lambda}{\Omega^2}$. This further implies $r(\hat{u}_0, \hat{v}_0) = \varpi(\hat{u}_0, \hat{v}_0) \pm \sqrt{\varpi(\hat{u}_0, \hat{v}_0)^2 - M^2}$ by solving r from (1.8) and hence $\varpi(\hat{u}_0, \hat{v}_0) \geq M$. Hence, $\varpi(\hat{u}_0, \hat{v}_0) = M$. Then plugging back to the solution of r above tells us that $r(\hat{u}_0, \hat{v}_0) = M$.

Since $\nu = \partial_u r < 0$ on $\hat{\mathcal{Q}}_{\max}$ (recall Lemma 2.3), r is strictly decreasing along direction of u, this in turn implies that $(\hat{u}_0, \hat{v}_0) \in \mathcal{H}^+$. Indeed, suppose that $\hat{u}_0 > \hat{u}_{\mathcal{H}^+}$, then $M < r(\hat{u}_{\mathcal{H}^+}, \hat{v}_0) < r(\hat{u}_{\mathcal{H}^+}, \hat{v}) \to M$ as $\hat{v} \to \infty$, which is a contradiction. On the other hand, $\hat{u}_0 < \hat{u}_{\mathcal{H}^+}$ is obviously false since $\lambda > 0$ before $\mathcal{H}^+$ (monotonicity in Raychaudhuri). Thus, it follows that $(\hat{u}_0, \hat{v}_0) \in \mathcal{H}^+$.

Moreover, from $\partial_v r \geq 0$ and $r(\hat{u}_{\mathcal{H}^+}, \hat{v}_0) = M = \lim_{\hat{v}\to\infty} r(\hat{u}_{\mathcal{H}^+}, \hat{v})$, we deduce that $r(\hat{u}_{\mathcal{H}^+}, \hat{v}) = M$ for all $\hat{v} \geq \hat{v}_0$. Hence, $\{\hat{u}_{\mathcal{H}^+}\} \times [\hat{v}_0, \infty) \subset \mathcal{A}$ and hence $\{\hat{u}_{\mathcal{H}^+}\} \times [\hat{v}_0, \infty) \subset \mathcal{A}'$.

By Raychaudhuri's equation (1.6), it follows that $\partial_v \phi$ vanishes identically on $\{\hat{u}_{\mathcal{H}^+}\} \times [\hat{v}_0, \infty)$. Since ϕ decays pointwisely along $\mathcal{H}^+$, ϕ itself vanishes on $\{\hat{u}_{\mathcal{H}^+}\} \times [\hat{v}_0, \infty)$.

Proof that $\lambda > 0$ behind $\mathcal{H}^+$: Suppose $\lambda(\hat{u}_0, \hat{v}_0) \leq 0$. Then one needs a Taylor expansion of $\partial_v r$ in u along $\mathcal{H}^+$. The proof goes similarly as the previous one. See [AKU24, Section 8.3.5].

[3] This hypothesis could be empty.

3.4. **Event horizon and eschatological gauge.**

Lemma 3.5. *There exists a constant $\theta \in (0,1)$ such that for ϵ sufficiently small and $\tau_f \in \mathfrak{B}(\mathcal{S}_0, \epsilon, A_0)$, it holds that $\Gamma^{\hat{u}}(\tau_f) \leq \theta U_*$.*

Proof. Since $\partial_{\hat{u}} r = -1$ on $\underline{C}_{\rm in}$ and $r(0,0) = \Lambda$, we see that $r(\Gamma^{\hat{u}}(\tau_f), 0) = \Lambda - \Gamma^{\hat{u}}(\tau_f)$. Using the bootstrap assumption of $\tilde{r}$, we then estimate

$$\Gamma^{\hat{u}}(\tau_f) = \Lambda - r(\Gamma^{\hat{u}}(\tau_f), 0) = \Lambda - \bar{r}_{\tau_f} \circ \Phi_{\tau_f}(\Gamma^{\hat{u}}(\tau_f), 0) - \tilde{r} \circ \Phi_{\tau_f}(\Gamma^{\hat{u}}(\tau_f), 0) = 99 M_0 + O(\epsilon),$$

which is quantitatively strictly smaller than $U_* = \frac{995}{10} M_0$ for ϵ sufficiently small. □

We first record the existence (definition) of the *eschatological gauge* Φ_∞, which is well-defined and C^1 on $\hat{\mathcal{D}}_\infty$:

Proposition 3.6. *For $(\hat{u}, \hat{v}) \in \hat{\mathcal{D}}_\infty$, the limit*

$$\Phi_\infty(\hat{u}, \hat{v}) := \lim_{\tau_f \to \infty} \Phi_{\tau_f}(\hat{u}, \hat{v})$$

exists and defines a C^1 diffeomorphism $\Phi_\infty : \hat{\mathcal{D}}_\infty \to [0, \infty) \times [0, \infty)$.

First, we observe the following immediate consequence of Lemma 3.5:

Lemma 3.7. *The limit $\hat{u}_{\mathcal{H}^+} := \lim_{\tau \to \infty} \Gamma^{\hat{u}}(\tau)$ exists and satisfies $\hat{u}_{\mathcal{H}^+} \leq \theta U_*$, where $\theta \in (0,1)$ is the constant from Lemma 3.5. Furthermore, $\hat{\mathcal{D}}_\infty = [0, \hat{u}_{\mathcal{H}^+}) \times [0, \infty)$.*

3.5. **Estimates of geometric quantities.** We only record one basic estimate and claim that the estimates of geometric quantities would require to use the gauge renormalization to help us integrate to get to the quantity at later times.

Lemma 3.8. *For any $A \geq 1$, ϵ sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\tfrac{1}{2} M \leq \varpi \leq 2M,$$
$$\tfrac{1}{2} \leq \frac{\nu}{\bar{\nu}} \leq 2,$$
$$\lambda > 0$$

in $\mathcal{D}_{\tau_f}$ and

$$1 - \mu \geq \tfrac{3}{4},$$
$$\lambda \geq \tfrac{1}{2},$$
$$-2 \leq \nu \leq -\tfrac{1}{2}, \tag{3.6}$$

in $\mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$.

References

[AAG17] Yannis Angelopoulos, Stefanos Aretakis, and Dejan Gajic. The trapping effect on degenerate horizons. *Ann. Henri Poincaré*, 18(5):1593–1633, 2017.

[AAG20] Y. Angelopoulos, S. Aretakis, and D. Gajic. Late-time asymptotics for the wave equation on extremal Reissner-Nordström backgrounds. *Adv. Math.*, 375:107363, 139, 2020.

[AKU24] Yannis Angelopoulos, Christoph Kehle, and Ryan Unger. Nonlinear stability of extremal reissner-nordström black holes in spherical symmetry, 2024.

[Are18] Stefanos Aretakis. *Dynamics of extremal black holes*, volume 33 of *SpringerBriefs in Mathematical Physics*. Springer, Cham, 2018.

[CT84] W. E. Couch and R. J. Torrence. Conformal invariance under spatial inversion of extreme Reissner-Nordström black holes. *Gen. Relativity Gravitation*, 16(8):789–792, 1984.

[Daf05a] Mihalis Dafermos. The interior of charged black holes and the problem of uniqueness in general relativity. *Comm. Pure Appl. Math.*, 58(4):445–504, 2005.

[Daf05b] Mihalis Dafermos. On naked singularities and the collapse of self-gravitating Higgs fields. *Adv. Theor. Math. Phys.*, 9(4):575–591, 2005.

[Daf05c] Mihalis Dafermos. Spherically symmetric spacetimes with a trapped surface. *Classical Quantum Gravity*, 22(11):2221–2232, 2005.

[DHRT21] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski, and Martin Taylor. The non-linear stability of the schwarzschild family of black holes, 2021.

[DHRT24] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski, and Martin Taylor. Quasilinear wave equations on asymptotically flat spacetimes with applications to kerr black holes, 2024.

[LO19a] Jonathan Luk and Sung-Jin Oh. Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. The interior of the black hole region. *Ann. of Math. (2)*, 190(1):1–111, 2019.

[LO19b] Jonathan Luk and Sung-Jin Oh. Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data II: the exterior of the black hole region. *Ann. PDE*, 5(1):Paper No. 6, 194, 2019.

[Tan24] Ning Tang. Stability of the catenoid for the hyperbolic vanishing mean curvature equation in 4 spatial dimensions, 2024.

[Ton] David Tong. Lectures on general relativity - 6. black holes.

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA
Email address: `ning_tang@math.berkeley.edu`