INTEGRATED LOCAL ENERGY DECAY ESTIMATES

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Contents

1. Introduction

We consider the linear wave equation on $\mathbb{R}^{1+d}$:

$$\begin{cases} -\partial_t^2 u + \Delta u = f, \\ (u, \partial_t u)(0) = (u_0, u_1). \end{cases} \tag{1.1}$$

For nice u, by multiplying $-\partial_t u$ on both sides of (1.1) and integrating by parts, we obtain

$$\|\nabla_{t,x} u(T)\|_{L_x^2}^2 \le \|\nabla_{t,x} u(0)\|_{L_x^2}^2 + |\langle \partial_t u, f\rangle_{L_t^2 L_x^2}| \tag{1.2}$$

and hence

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \le \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L_t^1 L_x^2}, \tag{1.3}$$

which is called the *uniform boundedness of energy.*

Remark 1.1. For $d \ge 3$, this inequality can be made rigorous by simply noticing that $\dot{H}^1 \subset L^{\frac{2d}{d-2}}$ and the definition that $\mathcal{D} \subset \dot{H}^1$ is dense. Then for compactly supported initial data, we can formulate the question by examining the energy estimates for $\Box(\chi_n(x)u) = \chi_n(x)f + \nabla\chi_n \cdot \nabla u + u\Delta\chi_n$, where $\chi_n(x) = \chi(x/n)$ and then take the limit. Moreover, for $(u_0, u_1) \in \dot{H}^1 \times L^2$, we can define the solution just by defining it to be the limit of u_m with initial data $(u_{0,m}, u_{1,m})$ giving by the energy inequality. With the embedding $\dot{H}^1 \subset L^{\frac{2d}{d-2}}$ in mind, the limit $u(t, \cdot)$ of u_m is a well-defined function in $\dot{H}_x^1$.

For $d = 1, 2$, one may just choose the initial data in $H^1 \times L^2$ or consider the constants produced by homogeneous norm carefully.

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These notes are taken for the PDE learning seminar in Spring 2023 introducing the integrated local energy decay estimates (ILED) for wave equation and a spectral theoretic characterization of the ILED. Thanks Sung-Jin Oh and Ovidiu-Neculai Avadanei for the original version of a set of notes concerning about the ILED..

2. Sharp integrated local energy decay estimate for the wave equation

Set

$$\|u\|_{LE} := \sup_{j>0} \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2L^2(\mathbb{R}_t\times A_j)}, \quad \|f\|_{LE^*} := \sum_{j>0} \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2L^2(\mathbb{R}_t\times A_j)},$$

where

$$A_j := \begin{cases} \{x : 2^{j-1} \le |x| < 2^j\}, & j \ge 1, \\ \{x : |x| \le 1\}, & j = 0. \end{cases}$$

We prove the following integrated local energy decay estimates in $\mathbb{R}^{1+d}$.

Theorem 2.1. *For $d \ge 3$, the solution u to linear wave equation (1.1) satisfies*

$$\|\nabla_{t,x} u\|_{LE} + \|r^{-1}u\|_{LE} \lesssim \|(u_0, u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{LE^*}. \tag{2.1}$$

Proof. **Step 1 :** We claim it suffices to establish the simpler bound

$$\begin{aligned} \|\nabla_{t,x} u\|^2_{L^2_tL^2_x([0,T]\times B_1)} &+ \|r^{-1}u\|^2_{L^2_tL^2_x([0,T]\times B_1)} \\ &\lesssim \|\nabla_{t,x}u(T)\|^2_{L^2_x} + \|\nabla_{t,x}u(0)\|^2_{L^2_x} + \left|\langle \beta(r)\partial_r u + \frac{\gamma(r)}{r}u, f\rangle_{L^2_tL^2_x}\right| \end{aligned} \tag{2.2}$$

for some $\beta, \gamma \in L^\infty$. Assuming this, the result follows from considering the scaled function $u^k = u(2^k t, 2^k x)$, which solves $\Box u^k = 2^{2k} f^k$ and combining with the energy estimates (1.2).

Specifically, due to $2^{-k}A_k \subset B_1$, (2.2) implies

$$\begin{aligned} &\|\nabla_{t,x} u^k\|^2_{L^2_tL^2_x([0,T]\times 2^{-k}A_k)} + \|r^{-1}u^k\|^2_{L^2_tL^2_x([0,T]\times 2^{-k}A_k)} \\ \lesssim &\|\nabla_{t,x}u^k(T)\|^2_{L^2_x} + \|\nabla_{t,x}u^k(0)\|^2_{L^2_x} + \left|\langle \beta(r)\partial_r u^k + \frac{\gamma(r)}{r}u^k, 2^{2k}f^k\rangle_{L^2_tL^2_x}\right|. \end{aligned}$$

Furthermore, thanks to scaling, this implies

$$\begin{aligned} &2^{-k}\|\nabla_{t,x} u\|^2_{L^2_tL^2_x([0,2^kT]\times A_k)} + 2^{-k}\|r^{-1}u\|^2_{L^2_tL^2_x([0,2^kT]\times A_k)} \\ \lesssim &\|\nabla_{t,x}u(2^kT)\|^2_{L^2_x} + \|\nabla_{t,x}u(0)\|^2_{L^2_x} + \int_{\mathbb{R}}\int_{\mathbb{R}^{1+d}} |\nabla_{t,x}u||f| + |r^{-1}u||f|\,dx\,dt. \end{aligned}$$

Combining with the energy estimates (1.2)

$$\|\nabla_{t,x}u(T)\|^2_{L^2_x} \le \|\nabla_{t,x}u(0)\|^2_{L^2_x} + |\langle \partial_t u, f\rangle_{L^2_tL^2_x}|,$$

we get

$$\begin{aligned} 2^{-k}\|\nabla_{t,x} u\|^2_{L^2_tL^2_x([0,2^kT]\times A_k)} &+ 2^{-k}\|r^{-1}u\|^2_{L^2_tL^2_x([0,2^kT]\times A_k)} \\ &\lesssim \|\nabla_{t,x}u(0)\|^2_{L^2_x} + \int_{\mathbb{R}}\int_{\mathbb{R}^{1+d}} |\nabla_{t,x}u||f| + |r^{-1}u||f|\,dx\,dt. \end{aligned} \tag{2.3}$$

Now we apply the Cauchy inequality $2ab \le \delta a^2 + \delta^{-1}b^2$, and then (2.1) follows by taking the supremum with respect to T and $k \ge 0$.

Step 2 : Set $X = \varphi(r)x^j\partial_j = r\varphi(r)\partial_r$ with $\beta(r) = r\varphi(r) \in L^\infty$. In order to prove (2.2), we examine these following to terms

$$\langle Xu, f\rangle, \qquad \langle \frac{\gamma(r)}{r}, f\rangle$$

with φ and γ to be determined.

Step 3 : Going forward, $\langle\cdot,\cdot\rangle$ denotes $\langle\cdot,\cdot\rangle_{L^2_x}$. We compute

$$\langle Xu, \Delta u\rangle = -\int u\partial_j\left(x^j\varphi(r)\Delta u\right)\,dx = \dots,$$

which gives

$$2\langle Xu, \Delta u\rangle = \langle[\Delta, X]u, u\rangle - \langle(d\varphi(r) + r\varphi'(r))u, \Delta u\rangle. \tag{2.4}$$

On the other hand, we compute

$$\begin{aligned}\Delta Xu = \Delta(r\varphi(r)\partial_r u) &= \Delta\varphi\cdot(r\partial_r u) + 2\nabla\varphi(r)\cdot\nabla(r\partial_r u) + \varphi(r)\Delta(r\partial_r u) = \dots\\ &= \left(\varphi'' + \frac{d+1}{r}\varphi'\right)r\partial_r u + 2\varphi' x^k\partial_k\partial_r u + \varphi(r)\Delta(r\partial_r u)\end{aligned}$$

and

$$X\Delta u = \varphi(r)x^j\Delta\partial_j u = \varphi(r)\Delta(x^j\partial_j u) - \dots = \varphi(r)\Delta(r\partial_r u) - 2\varphi(r)\Delta u.$$

Then by writing $[\Delta, X]u$ explicitly, we have

$$[\Delta, X]u = \left(\varphi'' + \frac{d+1}{r}\varphi'\right)r\partial_r u + 2\varphi'(r)r\partial_r^2 u + 2\varphi(r)\Delta u,$$

which implies

$$\langle[\Delta, X]u, u\rangle = -\langle\varphi'' r\partial_r u, u\rangle - (d+1)\langle\varphi'\partial_r u, u\rangle - 2\int r\varphi'(r)|\partial_r u|^2\,dx - 2\int\varphi(r)|\nabla u|^2\,dx.$$

On the other hand,

$$-\langle(d\varphi(r)+r\varphi'(r))u, \Delta u\rangle = (d+1)\varphi'\partial_r u, u + \langle\varphi'' r\partial_r u, u\rangle + d\int\varphi(r)|\nabla u|^2\,dx + \int r\varphi'(r)|\nabla u|^2.$$

Combining these two with (2.5), we get a magic cancellation of bad terms, which gives

$$2\langle Xu, \Delta u\rangle = -2\int r\varphi'(r)|\partial_r u|^2\,dx + \int r\varphi'(r)|\nabla u|^2\,dx + (d-2)\int\varphi(r)|\nabla u|^2\,dx. \tag{2.5}$$

By noticing

$$\langle Xh, h\rangle = -\langle\frac{d\varphi(r) + r\varphi'(r)}{2}h, h\rangle,$$

it follows from a direct integration by parts that

$$\int_0^T\langle Xu, -\partial_t^2 u\rangle = \langle Xu, -\partial_t u\rangle|_0^T - \int_0^T\langle\frac{d\varphi(r) + r\varphi'(r)}{2}\partial_t u, \partial_t u\rangle\,dt. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^d}\frac{d\varphi(r) + r\varphi'(r)}{2}|\partial_t u|^2\,dx\,dt - \int_0^T\int_{\mathbb{R}^d} r\varphi'(r)|\partial_r u|^2\,dx\,dt\\ &+\int_0^T\int_{\mathbb{R}^d}\left(\frac{1}{2}r\varphi'(r) + \frac{d-2}{2}\varphi(r)\right)|\nabla u|^2\,dx\,dt = \langle Xu, \partial_t u\rangle|_0^T + \int_0^T\langle Xu, f\rangle\,dt.\end{aligned} \tag{2.7}$$

Step 4 : First, we write

$$\int_0^T \langle \frac{\gamma(r)}{r}u, -\partial_t^2 u\rangle\, dt = \langle \frac{\gamma(r)}{r}u, -\partial_t u\rangle|_0^T + \int_0^T \int_{\mathbb{R}^d} \frac{\gamma(r)}{r}|\partial_t u|^2\, dx\, dt.$$

Now we compute

$$\langle \frac{\gamma(r)}{r}u, \Delta u\rangle = -\int \frac{\gamma(r)}{r}|\nabla u|^2\, dx - \int \frac{\gamma'(r)}{r}u\partial_r u\, dx + \int \frac{\gamma(r)}{r}u\partial_r u\, dx.$$

Then by IBP, we have

$$\int \frac{\gamma'(r)}{r}u\partial_r u\, dx = -\frac{d-2}{2}\int \frac{\gamma'(r)}{r^2}|u|^2\, dx - \frac{1}{2}\int \frac{\gamma''(r)}{r}|u|^2\, dx$$

and

$$\int \frac{\gamma(r)}{r^2}u\partial_r u\, dx = -\frac{d-3}{2}\int \frac{\gamma(r)}{r^3}|u|^2\, dx - \frac{1}{2}\int \frac{\gamma'(r)}{r^2}|u|^2\, dx.$$

These imply

$$\begin{aligned}\int_0^T \langle \frac{\gamma(r)}{r}u, f\rangle\, dt = \langle \frac{\gamma(r)}{r}u, -\partial_t u\rangle|_0^T + \int_0^T \int_{\mathbb{R}^d} \frac{\gamma(r)}{r}|\partial_t u|^2\, dx\, dt - \int_0^T \int \frac{\gamma(r)}{r}|\nabla u|^2\, dx\, dt \\ + \int_0^T \int_{\mathbb{R}^d} \left(\frac{d-3}{2}\frac{\gamma'(r)}{r^2} - \frac{d-3}{2}\frac{\gamma(r)}{r^3} + \frac{1}{2}\frac{\gamma''(r)}{r}\right)|u|^2\, dx\, dt.\end{aligned} \tag{2.8}$$

Step 5 : From (2.7) and (2.8), we obtain

$$\begin{aligned}-\int_0^T \int_{\mathbb{R}^d} \frac{d\varphi(r) + r\varphi'(r)}{2}|\partial_t u|^2\, dx\, dt + \int_0^T \int_{\mathbb{R}^d} \frac{(d-2)\varphi(r) - r\varphi'(r)}{2}|\nabla u|^2\, dx\, dt \\ + \int_0^T \int_{\mathbb{R}^d} \frac{\gamma(r)}{r}|\partial_t u|^2\, dx\, dt - \int_0^T \int \frac{\gamma(r)}{r}|\nabla u|^2\, dx\, dt \\ + \int_0^T \int_{\mathbb{R}^d} \left(\frac{d-3}{2}\gamma'(r) - \frac{d-3}{2}\frac{\gamma(r)}{r} + \frac{1}{2}r\gamma''(r)\right)|r^{-1}u|^2\, dx\, dt \\ \lesssim \langle Xu + \frac{\gamma(r)}{r}u, \partial_t u\rangle|_0^T + \int_0^T \langle Xu + \frac{\gamma(r)}{r}u, f\rangle\, dt.\end{aligned} \tag{2.9}$$

We choose $\varphi(r) = \frac{\beta(r)}{r}$ and $\gamma(r) = \frac{d-1}{2}\beta(r)$, then

$$\frac{\gamma(r)}{r} - \frac{d\varphi(r) + r\varphi'(r)}{2} = -\frac{\gamma(r)}{r} + \frac{(d-2)\varphi(r) - r\varphi'(r)}{2} = -\frac{\beta'(r)}{2}.$$

Now what we need is to choose $\beta \in L^\infty$ so that

$$\beta'(r) \geq \delta > 0, \quad \frac{d-3}{2}\beta'(r) - \frac{d-3}{2}\frac{\beta(r)}{r} + \frac{1}{2}r\beta''(r) < -\delta < 0 \tag{2.10}$$

for all $|r| \leq 1$ and

$$\beta'(r) \geq 0, \quad \frac{d-3}{2}\beta'(r) - \frac{d-3}{2}\frac{\beta(r)}{r} + \frac{1}{2}r\beta''(r) \leq 0 \tag{2.11}$$

for all $r \in [0, \infty)$.

For $d \geq 4$, we simply choose $\beta(r) = r + C$ when $|r| \leq 1$ for sufficiently large C since $v = r$ solves $v' - \frac{v}{r}$. Then β satisfies (2.10) and is positive, so one can then require β slowly grows to a positive constant function and hence (2.11) holds as well.

For $d = 3$, we choose $\beta(r) = -r\ln(r) + Cr$ when $|r| \leq 1$, so that $\beta'(r) = -(1 + \log(r)) + C$ and $r\beta''(r) = 1$ where $C > 1$ is a sufficiently large constant. Moreover, for all $r \in \mathbb{R}^+$, we require $\beta'(r) \leq 0$, $\beta''(r) \geq 0$ and $\beta \in L^\infty$, which can be easily made to hold. Then (2.9) implies (2.2), which completes the proof. □

Remark 2.2. The integrated local energy decay estimates (2.1) fail when $d = 1, 2$. However, (2.1) still true if we drop the term $\|r^{-1}u\|_{LE}$ on the left hand side. We construct an example explicitly in one dimension. We put

$$u_{0,n} = \chi(\frac{x}{n}),\ u_1 = 0,\ f = 0,$$

where $\chi \in C_c^\infty(\mathbb{R})$ with $\chi = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $0 \leq \chi \leq 1$.

It is easy to notice that

$$\|u_0\|_{\dot{H}^1}^2 = \frac{1}{n}\int |\chi'|^2\, dx.$$

We can use the d'Alembert's formula to write out the solution explicitly, which is

$$u(x,t) = \frac{1}{2}(u_0(x-t) + u_0(x+t)),$$

and therefore, it is easy to see

$$\begin{aligned}\|r^{-1}u\|_{LE} \geq & \int_0^\infty \int_0^1 \frac{|u_0(r-t) + u_0(r+t)|^2}{(1+r)^3}\, dr\, dt \geq n^2 \int_0^\infty \int_0^{\frac{1}{n}} \frac{|\chi(r-t) + \chi(r+t)|^2}{(1+nr)^3}\, dr\, dt \\ \gtrsim & n^2 \int_0^{1-\frac{1}{n}} \int_0^{\frac{1}{n}} \frac{1}{(1+nr)^3}\, dr\, dt \gtrsim n(1 - \frac{1}{n}),\end{aligned}$$

which means that the inequality would fail as n goes to infinity.

Moreover, one can predict this failure by the following heuristics.

Remark 2.3. The integrated local energy decay estimates are useful when the initial data does not have decay as $x \to \infty$. For initial data with sufficient decay, one can use the r^p-method. Note that even for nice initial data, sometimes we still prove ILED first as an intermediate result and then use the r^p-method.

Corollary 2.4. *Every solution of* (1.1) *satisfies*

$$\|\nabla_{t,x}u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r\rangle^{-1}u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|\nabla_{t,x}u(0)\|_{L_x^2} + \|f\|_{L_t^1 L_x^2 + LE^*}. \tag{2.12}$$

Proof. From an intermediate estiamte (2.3) in the first step of the proof for Theorem 2.1, we have

$$\|\nabla_{t,x}u\|_{LE}^2 + \|\langle r\rangle^{-1}u\|_{LE}^2 \lesssim \|\nabla_{t,x}u(0)\|_{L_x^2}^2 + \int_{\mathbb{R}}\int_{\mathbb{R}^{1+d}} |\nabla_{t,x}u||f| + |r^{-1}u||f|\, dx\, dt.$$

Thanks to the energy estimates (1.2) and Hardy's inequality ($d \geq 3$),

$$\|\nabla_{t,x}u\|_{L_t^\infty L_x^2 \cap LE}^2 + \|\langle r\rangle^{-1}u\|_{L_t^\infty L_x^2 \cap LE}^2 \lesssim \|\nabla_{t,x}u(0)\|_{L_x^2}^2 + \int_{\mathbb{R}}\int_{\mathbb{R}^{1+d}} |\nabla_{t,x}u||f| + |r^{-1}u||f|\, dx\, dt.$$

Then the result follows from $2ab \le \delta a^2 + \delta^{-1} b^2$. ☐

We now consider the perturbed equation

$$\begin{cases} (-\partial_t^2 - L)u = f \\ u(0) = u_0, \quad \partial_t u(0) = u_1 \end{cases} \tag{2.13}$$

where $L = -\Delta + b^k \partial_k + c$ is a perturbation of the minus Laplacian which satisfies the decay condition

$$\sum_{j=0}^{\infty} \sup_{\mathbb{R}_t \times A_j} \langle x \rangle |b| + \langle x \rangle^2 |\partial_l b^l| + \langle x \rangle^2 |c| < \kappa, \tag{2.14}$$

where $\kappa > 0$ is a positive constant and b^j, c are smooth functions. At this point, we allow b^j, c to be complex-valued and time-dependent.

Corollary 2.5. *Let u be a solution of (2.13). If κ is small enough, then u satisfies*

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|\nabla_{t,x} u(0)\|_{L_x^2} + \|f\|_{L_t^1 L_x^2 + LE^*}.$$

Proof. We write $Bu = b^j \partial_j u + cu$ and rewrite the equation as $(-\partial_t^2 + \Delta)u = f - Bu$. Thus,

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|(u_0, u_1)\|_{\dot H^1 \times L^2} + \|f - Bu\|_{LE^*}$$

Furthermore,

$$\begin{aligned} \|Bu\|_{LE^*} \simeq \sum_{k=0}^{\infty} 2^{\frac{k}{2}} \|Bu\|_{L^2 L^2(\mathbb{R}_t \times A_k)} &= \sum_{k=0}^{\infty} 2^{\frac{k}{2}} \|b^l \partial_l u\|_{L^2 L^2(\mathbb{R}_t \times A_k)} + 2^{\frac{k}{2}} \|cu\|_{L^2 L^2(\mathbb{R}_t \times A_k)} \\ &\lesssim \kappa (\|\nabla_{t,x} u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE}) \end{aligned} \tag{2.15}$$

Thus, after combining these two inequalities, the result follows if κ is sufficiently small so that we can absorb it to the left hand side. ☐

3. Strichartz estimates

Definition 3.1. A pair (p, q) is said to be *wave-admissible* in dimension $d + 1$ if

$$p \in [2, \infty], \quad \frac{1}{p} + \frac{d-1}{2q} \le \frac{d-1}{4}, \quad (p, q, d) \neq (2, \infty, 3)$$

We begin by recalling the Strichartz estimates for $\Box$, which captures the dispersive property of finite energy solutions to the wave equation in a useful way.

Theorem 3.2. *Let $u_0, u_1 \in \mathscr{S}(\mathbb{R}^d)$, and let u be the solution to (1.1) with this initial data. Let (p, q) and $(\tilde p, \tilde q)$ be pairs of wave-admissible exponents, which also obey the scaling conditions*

$$\frac{d}{2} - 1 = \frac{1}{p} + \frac{d}{q} = \frac{1}{\tilde p'} + \frac{d}{\tilde q'} - 2, \tag{3.1}$$

where $\tilde p'$ and $\tilde q'$ are the Lebesgue duals to $\tilde p$ and $\tilde q$, i.e $\frac{1}{\tilde p'} + \frac{1}{\tilde p} = \frac{1}{\tilde q'} + \frac{1}{\tilde q} = 1$. Then, we have the following

$$\|\nabla_{t,x} u\|_{L^\infty L^2} + \|u\|_{L^p L^q} \lesssim_{p,q,\tilde p,\tilde q} \|(u_0, u_1)\|_{\dot H^1 \times L^2} + \|f\|_{L^{\tilde p'} L^{\tilde q'}}$$

We are now going to prove the following result:

Theorem 3.3 (Rodnianski-Schlag). *We assume that the coefficients of (2.13) satisfy (2.14) for some $\kappa > 0$ (in particular, κ can be large). We also assume that ILED (2.12) holds for the perturbated equation (2.13).*

Let $u_0, u_1 \in \mathscr{S}(\mathbb{R}^d)$, and let u be the solution to (2.13) with this initial data. Let (p, q) and $(\tilde{p}, \tilde{q})$ be pairs of wave-admissible exponents, which also obey the scaling conditions (3.1) In addition, we assume that they also satisfy the non-endpoint condition $p, \tilde{p} > 2$. Then,

$$\|\nabla_{t,x} u\|_{L^\infty L^2} + \|u\|_{L^p L^q} \lesssim_{p,q,\tilde{p},\tilde{q}} \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1_t L^2_x + LE^*}.$$

Remark 3.4. One can also add another term $\|\nabla_{t,x} u\|_{L^{p_1} L^{q_1}}$ in the left hand side, provided the stronger version of homogeneous Strichartz estimates, where (p_1, q_1) is another wave-admissible pair. To put $\|f\|_{L^{\tilde{p}'} L^{\tilde{q}'}}$, we need the ILED assumption on the dual problem.

Proof. We write $L = -\Delta + B$ and

$$(-\partial_t^2 + \Delta) u = Bu + f.$$

Thanks to the ILED assumption (2.12) for $-\partial_t^2 - L$, we know

$$\|Bu\|_{LE^*} \lesssim \kappa (\|\nabla_{t,x} u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE}) \lesssim \|(u_0, u_1)\|_{\dot{H}^1_x \times L^2_x} + \|f\|_{L^1_t L^2_x + LE^*}, \tag{3.2}$$

where we use the assumption for κ in (2.14) in the first inequality and the ILED for the perturbed equation $(-\partial_t^2 + L) u = f$ in the second in equality.

Moreover, if w solves the homogeneous problem with initial data $(0, u_0, u_1)$, then

$$\|\nabla_{t,x} w\|_{L^\infty_t L^2_x \cap LE} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2},$$

which means that we only need to prove the theorem for $u - w$, which is a solution to the inhomogeneous problem with zero initial data. In other words, it suffices to show

$$\|u\|_{L^p_t L^q_x} \lesssim_{p,q} \|F\|_{L^1_t L^2_x + LE^*}.$$

for forward solutions u to $(\partial_t^2 - \Delta) u = F$.

If v solves the homogeneous problem, then

$$\|\nabla_{t,x} v\|_{L^\infty_t L^2_x \cap LE} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2},$$

where v can be expressed in terms of the Duhamel's formula

$$v(t) = \cos(t\sqrt{-\Delta}) u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} u_1.$$

Therefore, it is equivalent to write

$$\left\| \cos(t\sqrt{-\Delta}) \nabla u_0 \right\|_{L^\infty_t L^2_x \cap LE} \lesssim \|\nabla u_0\|_{L^2}, \quad \left\| \nabla \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} u_1 \right\|_{L^\infty_t L^2_x \cap LE} \lesssim \|u_1\|_{L^2}.$$

Then by duality,

$$\left\| \int_{-\infty}^{\infty} \cos(t\sqrt{-\Delta}) F(t)\, dt \right\|_{L^2_x} \lesssim \|F\|_{L^1_t L^2_x + LE^*}, \quad \left\| \int_{-\infty}^{\infty} \sin(t\sqrt{-\Delta}) F(t)\, dt \right\|_{L^2_x} \lesssim \|F\|_{L^1_t L^2_x + LE^*}.$$

Moreover, the homogeneous Strichartz estimates imply

$$\left\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\int_{-\infty}^{\infty}\cos(s\sqrt{-\Delta})F(s)\,ds\right\|_{L_t^pL_x^q}\lesssim\left\|\int_{-\infty}^{\infty}\cos(s\sqrt{-\Delta})F(s)\,ds\right\|_{L_x^2}$$

$$\left\|\cos(t\sqrt{-\Delta})\int_{-\infty}^{\infty}\frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s)\,dt\right\|_{L_t^pL_x^q}\lesssim\left\|\nabla\int_{-\infty}^{\infty}\frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L_x^2}.$$

Combining the estimates above,

$$\left\|\int_{-\infty}^{\infty}\frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L_t^pL_x^q}\lesssim\|F\|_{L_t^1L_x^2+LE^*}.$$

Thanks to Christ-Kiselev lemma, we have

$$\|u\|_{L_t^pL_x^q}\lesssim_{p,q}\|F\|_{L_t^1L_x^2+LE^*}.$$

for forward solutions u to $(\partial_t^2-\Delta)u=F$, which completes the proof. □

Remark 3.5. In general, Strichartz estimates are difficult to prove. The advantage of this is to reduce the proof of Strichartz estimates for $-\partial_t^2-L$ to the ILED for $-\partial_t^2-L$ by invoking the Strichartz estimates for $\Box$.

4. Spectral theoretic characterization of local energy decay

We assume L is a linear operator of the form $L=-\Delta+b^k\partial_k+c$, where b,c are time-independent, b is purely imaginary and divergence free and c is real. Moreover, L obeys the decay condition (2.14) for some (possibly large) $\kappa>0$. With these conditions, it follows that L is self-adjoint with $\mathcal{D}(L)=\{u\in L^2 : Pu\in L^2\}$ and hence $\sigma(L)\subset\mathbb{R}$. (The assumptions here are natural in many physics models such as magnetic potentials.)

We consider the Cauchy problem

$$-\partial_t^2u-Lu=f,\quad (u,\partial_tu)(0)=(u_0,u_1)\in\dot{H}^1\times L^2 \tag{4.1}$$

and set the energy to be

$$E[u](t):=\frac{1}{2}\|\partial_tu\|_{L^2}^2+\frac{1}{2}\langle Lu,u\rangle.$$

Then $\partial_tE[u](t)=-\langle f,\partial_tu\rangle$ and this implies that

$$E[u](t)\le E[u](0)+\int\langle f,\partial_tu\rangle\,dt.$$

Therefore, if we assume the coercivity condition

$$\langle Lu,u\rangle\ge\|u\|_{\dot{H}^1}^2, \tag{4.2}$$

then by combining with the obvious bound $\langle Lu,u\rangle\le\|u\|_{\dot{H}^1}^2$ for nice u, we obtain the uniform boundedness of energy as a consequence

$$\|\nabla_{t,x}u\|_{L_t^\infty L_x^2}\le\|(u_0,u_1)\|_{\dot{H}^1\times L^2}+\|f\|_{L_t^1L_x^2}.$$

We remark at this point that (4.2) rules out the existence of negative eigenvalues by simply noticing $\langle\lambda_0u_0,u_0\rangle\ge0$ for any (u_0,λ_0) such that $Lu_0=\lambda_0u_0$. Indeed, any negative eigenvalue of L would lead to solutions to (4.1) that grows exponentially as either $t \to \infty$ or $t \to -\infty$, which is a clear violation of the uniform boundedness of energy.

Our goal is to give a characterization of the ILED in terms of spectral theory. Such a connection shows how tools from spectral theory can be employed to establish ILED. For this purpose, we begin by defining the (wave) resolvents of L as

$$R_z := (z^2 - L)^{-1}, \tag{4.3}$$

which are well-defined as an operator $L^2 \to \mathcal{D}(L)$ if $z^2 \notin \sigma(L)$.

Remark 4.1. In fact, we do not care about what $\mathcal{D}(L)$ is too much. We only need L to be self-adjoint. Moreover, one can do a change of variable $\tau = z^2$ to turn R_z into standard resolvents. The choice of z^2 here is more natural due to the Fourier transform in time we would employ later.

We introduce the spatial counterparts of the norms LE and LE^*, namely,

$$\|u\|_{\mathcal{LE}} = \sup_{j\geq 0} \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(A_j)}, \quad \|f\|_{\mathcal{LE}^*} = \sum_{j\geq 0} \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(A_j)}.$$

Theorem 4.2. *The following statements are equivalent :*

(1) Every solution u to (2.13) obeys (2.1).
(2) For every $\tau \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$|\tau \mp i\varepsilon| \|R_{\tau\mp i\varepsilon} g\|_{\mathcal{LE}} + \|\nabla_x R_{\tau\mp i\varepsilon} g\|_{\mathcal{LE}} + \|\langle r\rangle^{-1} R_{\tau\mp i\varepsilon} g\|_{\mathcal{LE}} \lesssim \|g\|_{\mathcal{LE}^*}. \tag{4.4}$$

Remark 4.3. The basic idea is to take the Fourier transform in time, then we formally obtain $(\tau^2 - L)\widehat{u}(\tau) = \widehat{f}(\tau)$. Although $f \in LE^* \subset L^2(\mathbb{R}^{1+d})$, a finite energy solution u might not be square integrable in time. To carry out this strategy in a rigorous fashion, we need the following reductions.

Proof. Given $f \in LE^*$ with $\mathrm{supp}(\cdot, x)$ away from $\{t = -\infty\}$, we define a forward solution to be the one such that $u(t) \to 0$ in $\dot{H}^1 \times L^2$ as $t \to -\infty$. For $f \in \mathscr{S}(\mathbb{R}^{1+d})$, the forward solution is given by Duhamel's formula

$$u(t) = \int_{-\infty}^{t} \frac{\sin(t-s)\sqrt{L}}{\sqrt{L}} f(s)\, ds,$$

where $\frac{\sin(t-s)\sqrt{L}}{\sqrt{L}}$ can be viewed as a formal symbol denoting the solution for homogeneous equation with initial data $(0, f(s))$. One can also make sense of this using spectral calculus.

Step 1 : Reduction to forward solutions. In this step, our goal is to show that the following two statements are equivalent :

(a) For any $f \in \mathscr{S}(\mathbb{R}^{1+d})$, the solution u corresponds to f with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, (2.1) holds.
(b) For any $f \in \mathscr{S}(\mathbb{R}^{1+d})$, the forward solution u corresponds to f, we have

$$\|\nabla_{t,x} u\|_{LE} + \|r^{-1} u\|_{LE} \lesssim \|f\|_{LE^*}. \tag{4.5}$$

It is obvious that (a) implies (b) by choosing the initial time to tend to $-\infty$. For the converse, we assume (b) is true and use an argument similar to the Rodnianski-Schlag argument. Let v be the solution to the free wave equation (1.1) with the same initial data (f, u_0, u_1), then clearly, v satisfies

$$\|\nabla_{t,x} v\|_{LE} + \|r^{-1} v\|_{LE} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*}.$$

Moreover, we write

$$(-\partial_t^2 - L)(u - v) = Bv$$

and we may estimate

$$\|Bv\|_{LE^*} \lesssim \kappa(\|\nabla_{t,x} v\|_{LE} + \|\langle r\rangle^{-1} v\|_{LE}) \lesssim \|(u_0, u_1)\|_{\dot{H}^1_x \times L^2_x} + \|f\|_{L^1_t L^2_x + LE^*},$$

where we do not need the smallness of κ, in contrast with (3.2).

Let v_f, v_b be the forward and backward solutions corresponding to $1_{(0,\infty)}(t) \cdot Bv$ and $1_{(-\infty,0)}(t) \cdot Bv$, respectively. One can define the forward solutions in LE^* (at least those in LE^* with support in time away from $t = -\infty$) by extending (4.5) by density $\mathscr{S} \subset LE^*$, and hence we can replace f by $1_{(0,\infty)}(t) \cdot Bv$ with corresponding forward and backward solutions v_f in (4.5), that is,

$$\|\nabla_{t,x} v_f\|_{LE} + \|r^{-1} v_f\|_{LE} \lesssim \|1_{(-\infty,0)}(t) \cdot Bv\|_{LE^*} \lesssim \|Bv\|_{LE^*}.$$

Similarly,

$$\|\nabla_{t,x} v_b\|_{LE} + \|r^{-1} v_b\|_{LE} \lesssim \|Bv\|_{LE^*}$$

and therefore

$$\|\nabla_{t,x}(v + v_f + v_b)\|_{LE} + \|r^{-1}(v + v_f + v_b)\|_{LE} \lesssim \|(u_0, u_1)\|_{\dot{H}^1_x \times L^2_x} + \|f\|_{LE^*}.$$

Note that $u - v - v_f - v_b$ is a finite energy solution with zero data, it follows that $u = v + v_f + v_b$ by uniqueness, which proves (2.1).

Step 2 : Reduction to damped forward solutions. For any $f \in \mathscr{S}(\mathbb{R}^{1+d})$, the corresponding forward solution have adequate time decay as $t \to -\infty$ to apply the Plancherel theorem. However, their behavior as $t \to +\infty$ is still potentially problematic. In this step, we resolve this issue by showing that (b) is equivalent to

(c) For any $f \in \mathscr{S}(\mathbb{R}^{1+d})$, the forward solution u corresponds to f,

$$\|e^{-\varepsilon t} \nabla_{t,x} u\|_{LE} + \|e^{-\varepsilon t} r^{-1} u\|_{LE} \lesssim \|e^{-\varepsilon t} f\|_{LE^*} \tag{4.6}$$

for all $\varepsilon > 0$.

The implication from (c) to (b) : Thanks to (c), we have

$$2^{-j/2} \|e^{-\varepsilon t} \nabla_{t,x} u\|_{L^2 L^2(\mathbb{R}_t \times A_j)} + 2^{-j/2} \|e^{-\varepsilon t} r^{-1} u\|_{L^2 L^2(\mathbb{R}_t \times A_j)} \lesssim \|e^{-\varepsilon t} f\|_{LE^*}$$

and furthermore, for all K,

$$2^{-j/2} e^{-\varepsilon K} \|\nabla_{t,x} u\|_{L^2 L^2((-\infty,K] \times A_j)} + 2^{-j/2} e^{-\varepsilon K} \|r^{-1} u\|_{L^2 L^2((-\infty,K] \times A_j)} \lesssim \|e^{-\varepsilon t} f\|_{LE^*}.$$

Let $\varepsilon \to 0$ and then let $K \to \infty$, then (b) follows.

The implication from (b) to (c) : We compute

$$\|e^{-\varepsilon t}\nabla_{t,x}u\|_{LE}=\sup_{k\geq 0}\|\langle r\rangle^{-\frac12}e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_k)}\simeq\sup_{k\geq 0}\left(\|\langle r\rangle^{-\frac12}1_{(j,j+1]}(t)e^{-\varepsilon t}\nabla_{t,x}u\|^2_{L^2L^2(\mathbb{R}_t\times A_k)}\right)^{\frac12}$$
$$\leq\sup_{k\geq 0}\left(\sum_{j\in\mathbb{Z}}\left(e^{-\varepsilon j/2}\|\langle r\rangle^{-\frac12}1_{(j,j+1]}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_k)}\right)^2\right)^{\frac12}$$
$$=\left\|\left\|e^{-\varepsilon j/2}\|\langle r\rangle^{-\frac12}1_{(j,j+1]}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_k)}\right\|_{\ell^2_j}\right\|_{\ell^\infty_k}\lesssim\left(\sum_{j\in\mathbb{Z}}\left(e^{-\varepsilon j/2}\|1_{(j,j+1]}(t)\nabla_{t,x}u\|_{LE}\right)^2\right)^{\frac12},\tag{4.7}$$

where the last step follows from Minkowski's inequality. Similarly,

$$\|e^{-\varepsilon t}\langle r\rangle^{-1}u\|_{LE}\lesssim\left(\sum_{j\in\mathbb{Z}}\left(e^{-\varepsilon j/2}\|1_{(j,j+1]}(t)\langle r\rangle^{-1}u\|_{LE}\right)^2\right)^{\frac12}.$$

On the other hand,

$$\|e^{-\varepsilon t}f\|_{LE^*}=\sum_{k\geq 0}\|\langle r\rangle^{\frac12}e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A_k)}$$
$$\simeq\sum_{k\geq 0}\left(\sum_{j\in\mathbb{Z}}\left(e^{-\varepsilon j/2}\|1_{(j,j+1]}(t)\langle r\rangle^{1/2}f\|_{L^2L^2(\mathbb{R}_t\times A_k)}\right)^2\right)^{\frac12}\gtrsim\left(\sum_{j\in\mathbb{Z}}\left(e^{-\varepsilon j/2}\|1_{(j,j+1]}(t)f\|_{LE^*}\right)^2\right)^{\frac12}.$$

Therefore, it suffices to show

$$\|U_j\|_{\ell^2}\lesssim\|F_j\|_{\ell^2},\tag{4.8}$$

where

$$U_j:=e^{-\varepsilon j/2}\left(\|1_{(j,j+1]}(t)\nabla_{t,x}u\|_{LE}+\|1_{(j,j+1]}(t)\langle r\rangle^{-1}u\|_{LE}\right),\quad F_j:=e^{-\varepsilon j/2}\|1_{(j,j+1]}(t)f\|_{LE^*}.$$

To simplify our notations, set $f_j:=1_{(j,j+1]}(t)f$ and use u_j to denote the forward solution corresponded to f_j.

Note that

$$u_j(t)=\int_{-\infty}^t\frac{\sin(t-s)\sqrt{L}}{\sqrt{L}}f_j(s)\,ds,$$

we know $\mathrm{supp}u_j(\cdot,x)\subset[j,\infty)$ and therefore

$$1_{(j,j+1]}(t)u(t,x)=1_{(j,j+1]}(t)\sum_{j'<j}u_{j'}(t,x).$$

Obviously, such equality also holds by replacing u by $\nabla_{t,x}u$ and $\langle r\rangle^{-1}u$.

By (b), we compute

$$\begin{aligned}U_j =& e^{-\varepsilon j/2}\left(\|1_{(j,j+1]}(t)\nabla_{t,x}\sum_{j'<j}u_{j'}\|_{LE}+\|1_{(j,j+1]}(t)\langle r\rangle^{-1}\sum_{j'<j}u_{j'}\|_{LE}\right)\\ \lesssim& e^{-\varepsilon j/2}\left(\sum_{j'<j}\|1_{(j,j+1]}(t)\nabla_{t,x}u_{j'}\|_{LE}+\|1_{(j,j+1]}(t)\langle r\rangle^{-1}u_{j'}\|_{LE}\right)\\ \lesssim& e^{-\varepsilon j/2}\left(\sum_{j'<j}\|\nabla_{t,x}u_{j'}\|_{LE}+\|\langle r\rangle^{-1}u_{j'}\|_{LE}\right)\lesssim e^{-\varepsilon j/2}\sum_{j'<j}e^{\varepsilon j'/2}F_{j'}=\sum_{j'<j}e^{-\varepsilon(j-j')/2}F_{j'},\end{aligned}$$

where the first inequality follows in the same spirit of (4.7) and the third inequality follows from (b). Note that the convolution kernel $e^{-\varepsilon k/2}$ is integrable in $k\in\{k\in\mathbb{Z}:k>0\}$ for all $\varepsilon>0$, it follows from Schur's lemma or Young's inequality that (4.8) holds.

Step 3 : Reduction to a form for which Plancherel's theorem can be applied. Since LE and LE^* norm is not of the form $L^2_tX_x$ for some Banach space X, we want to reduce it to such a form.

We claim that (c) is equivalent to

(d) For any j,k and any f with support in A_k, we have

$$2^{-j/2}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_j)}+2^{-3j/2}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A_j)}\lesssim 2^{k/2}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A_k)}.$$

Let u^k be the forward solution corresponding to $\chi_k f$, then $u=\sum_k u^k$, where χ_k localize to $A_{k'}=A_k\cup A_{k+1}$ such that $\sum\chi_k=1$.

The implication from (c) to (d) is obvious thanks to the support property. The implication from (d) to (c) follows from the triangle inequality by the partition of unity.

Step 4 : Application of the Plancherel's theorem and closing the proof. Note that for nice f,u_0,u_1, the forward solution u given by existence theory is at least continuous in time so that we can take the Fourier transform after we multipliy it by $e^{-\varepsilon t}$. By taking a Fourier transform in time, (d) is equivalent to

$$2^{-j/2}\|(|\tau-i\varepsilon|,\nabla_x)\widehat{u}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A_j)}+2^{-j/2}\|\langle r\rangle^{-1}\widehat{u}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A_j)}\lesssim 2^{k/2}\|\widehat{f}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A_k)}.$$

By noticing that

$$\widehat{u}(\tau-i\varepsilon)=R_{\tau-i\varepsilon}\widehat{f}(\tau-i\varepsilon),$$

we know

$$\begin{aligned}2^{-j/2}\|(|\tau-i\varepsilon|,\nabla_x)R_{\tau-i\varepsilon}\widehat{f}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A_j)}+2^{-j/2}\|\langle r\rangle^{-1}R_{\tau-i\varepsilon}\widehat{f}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A_j)}\\ \lesssim 2^{k/2}\|\widehat{f}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A_k)}.\end{aligned}$$

In particular, by choosing $f(t,x)=\phi(t)g(x)$ with ϕ polynomially growing, smooth and support away from $-\infty$, we know the equivalence of (d) and the second statement in the theorem, which completes the proof. □

If we use the standard notation for resolvents, then the second statement implies that for any $\lambda>0$,

$$\sqrt{\lambda}\|R_{\lambda\pm i\varepsilon}g\|_{\mathcal{LE}}+\|\partial R_{\lambda\pm i\varepsilon}g\|_{\mathcal{LE}}\lesssim\|g\|_{\mathcal{LE}^*}. \tag{4.9}$$

Theorem 4.4. *Suppose that (4.9) holds for any $\lambda > 0$, then L has only purely absolutely continuous spectrum on any compact subinterval $[a, b] \subset (0, \infty)$.*

Proof. By density, it suffices to show that μ_f is absolutely continuous for any $f \in C_c^\infty$ on $[a, b]$. By Stone's formula, we have

$$\frac{1}{2}\left(\mu_f((a,b)) + \mu_f((a,b))\right) = \lim_{\varepsilon \to 0_+} \frac{1}{2\pi i} \int_a^b \langle R_{\lambda - i\varepsilon} f - R_{\lambda + i\varepsilon} f, f \rangle \, d\lambda.$$

First, take $a = b$, then $\mu_f(\{a\}) = 0$ for all a. Since $\langle R_{\lambda - i\varepsilon} f - R_{\lambda + i\varepsilon} f, f \rangle$ is uniformly bounded on $[a, b]$, uniformly in ε, dominated convergence theorem, we know

$$\mu_f\left((a,b)\right) = \frac{1}{2\pi i} \int_a^b \lim_{\varepsilon \to 0_+} \langle R_{\lambda - i\varepsilon} f - R_{\lambda + i\varepsilon} f, f \rangle \, d\lambda = \frac{1}{2\pi i} \int_a^b g(\lambda) \, d\lambda,$$

for some $g(\lambda)$. Note that the limits of $R_{\lambda - i\varepsilon} f$ and $R_{\lambda + i\varepsilon}$ are different, which is the limit absorption principle. This means that $d\mu_f \ll d\lambda$ and therefore it is absolutely continuous. □

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