MODE STABILITY FOR THE WAVE/TEUKOLSKY EQUATION ON KERR

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Overview

In this note, we mainly focus on understanding the work [Shl15], where the author proves quantitative mode stability for the wave equations with the subextremal Kerr background. An application of the quantitative mode stability result is to derive the ILED in bounded-frequency regime, which is the regime mentioned in [DR12, Section 11.7] but we will not dive into the details. Moreover, we mention some further generalizations to Teukolsky equations and extremal Kerr in Section 2.7. In Remark 1.9, one can find a comparison with the instability for Klein-Gordon on Kerr spacetime. For a survey of Kerr stability, one can refer to [Kla23].

Here is a guide for the reader who would like to know the main ideas effectively :

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1. Understanding the definition of mode solutions in Section 1.3 and how the boundary conditions are obtained.
2. Understanding a physical space method for the mode stability in Minkowski or Schwarzschild via Section 2.1. This tells us that ∂_t fails to be timelike in the ergoregion of Kerr is the main difficulty of mode stability of Kerr (this implies the nonexistence of nice conservation laws).
3. Understanding the proof of mode stability using microlocal energy current and how the current is motivated via Section 2.2.
4. In subsequent sections, we mimic the proof in the Schwarzschild case in Section 2.2 and find out the difficulty comes from the superradiant frequencies, which is the frequency manifestation of existence of ergoregion. Then one can refer to Remark 2.2 for the main idea of the proof for the mode stability of Kerr. Then in the last two subsections, we briefly mention some heuristics of the Whiting's integral transformation and further related works.

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1. Preliminaries

1.1. **Schwarzschild and Kerr spacetimes.**

1.1.1. *The Schwarzschild exterior.* Let $\mu = 2M/r$, then the Schwarzschild metric is given by

$$g = -(1-\mu)\, dt^2 + (1-\mu)^{-1}\, dr^2 + r^2 d\sigma_{\mathbb{S}^2}.$$

By introducing the Regge-Wheeler tortoise coordinate

$$r^*(r) = \int_{3M}^{r} (1-\mu)^{-1}\, dr = r + 2M\ln(r-2M) - 3M - 2M\ln M, \tag{1.1}$$

it can be written as

$$g = -(1-\mu)\, dt^2 + (1-\mu)\,(dr^*)^2 + r^2 d\sigma_{\mathbb{S}^2}.$$

The coordinate r^* is $+\infty$ at spatial and null infinity; $-\infty$ at the event horizon and 0 at $r = 3M$. The set $\{r = 3M\}$ is known as the photon sphere. On this set trapping occurs: there exist null geodesics that lie in this set. This suggests, via geometrical optics considerations, that one has to lose derivatives while proving energy estimates.

We also define the retarded and advanced Eddington-Finkelstein coordinates u and v by

$$u = \frac{1}{2}(t - r^*), v = \frac{1}{2}(t + r^*),$$

then in these coordinates, we have

$$g = -4(1-\mu)\, du\, dv + r^2 d\sigma_{\mathbb{S}^2}.$$

At the event horizon $\mathcal{H}^+$, $u = +\infty$. At future null infinity $\mathcal{I}^+$, $v = +\infty$.

1.1.2. *The Kerr metric.* Fix a pair of parameters (a, M) with $|a| < M$, and define $r_+ := M + \sqrt{M^2 - a^2}$. Define the underlying manifold M to be covered by a "Boyer-Lindquist" coordinate chart $(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$. The Kerr metric then takes the form

$$g_{a,M} = -(1 - \frac{2Mr}{\rho^2})\, dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\, dt\, d\phi + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\theta^2 + \sin^2\theta \frac{\Pi}{\rho^2}\, d\phi^2,$$
$$r_\pm := M \pm \sqrt{M^2 - a^2}, \quad \Delta := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2,$$
$$\rho^2 := r^2 + a^2\cos^2\theta, \quad \Pi := (r^2 + a^2)^2 - a^2\sin^2\theta\Delta.$$

As in the Schwarzschild case, it is convenient to define $r^*(r) : (r_+, \infty) \to (-\infty, \infty)$ coordinate up to a constant by

$$r^*(r) = \int \frac{r^2 + a^2}{\Delta}\, dr.$$

1.2. **Conservation laws in general spacetimes.** We follow [Luk10] to record some basics. For any Lorentz metric g, we consider the scalar wave equation $\Box_g \phi = F$, where ϕ is a real-valued smooth function. We define the energy-momentum tensor

$$T_{\mu\nu} = T_{\mu\nu}(\phi) := \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi,$$

then a direct computation yields

$$\nabla^\mu T_{\mu\nu} = F\partial_\nu\phi + \partial_\mu\phi\nabla^\mu\partial_\nu\phi - \partial^\alpha\phi\nabla_\nu\partial_\alpha\phi = F\partial_\nu\phi.$$

It then follows from the symmetry $\Gamma^\delta_{\mu\nu} = \Gamma^\delta_{\nu\mu}$ that

$$\nabla^\mu T_{\mu\nu} = F\partial_\nu\phi. \tag{1.2}$$

Given a vector field $V = V^\mu\partial_\mu$, we define the associated currents :

$$J^V_\mu(\phi) = V^\nu T_{\mu\nu}(\phi).$$

We compute the divergence of J^V :

$$\nabla^\mu J^V_\mu = (\nabla_\mu V_\nu)T^{\mu\nu} + V^\nu F\partial_\nu\phi = \pi^V_{\mu\nu}T^{\mu\nu} + V^\nu F\partial_\nu\phi,$$

where

$$\pi^V_{\mu\nu} := \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu)$$

is known as the deformation tensor. We also introduce the shorthand notation

$$K^V_\mu := \pi^V_{\mu\nu}T^{\mu\nu}, \quad \nabla^\mu J^V_\mu = K^V_\mu + F(V\phi).$$

In particular, both $\pi^V_{\mu\nu}$ and K^V_μ vanish if V is Killing.

Analogously, we define modified currents

$$J^{V,w}_\mu(\phi) := J^V_\mu(\phi) + \frac{1}{8}(w\partial_\mu\phi^2 - \phi^2\partial_\mu w), \quad K^{V,w}_\mu(\phi) := K^V_\mu + \frac{1}{4}w\partial^\nu\phi\partial_\nu\phi - \frac{1}{8}\phi^2\Box_g w,$$

and hence

$$\nabla^\mu J^{V,w}_\mu = K^V_\mu + F(V\phi) + \frac{1}{4}w\partial^\nu\phi\partial_\nu\phi + \frac{1}{4}w\phi F - \frac{1}{8}\phi^2\Box_g w = K^{V,w}_\mu(\phi) + F(V\phi) + \frac{1}{4}w\phi F.$$

We integrate by parts with this in a region $\mathcal{B}$ bounded to the future by Σ_1 and to the past by Σ_0. The region $\mathcal{B}$ should have no other boundary. Denoting the future-directed normal to Σ_0 and Σ_1 by $n^\mu_{\Sigma_0}$ and $n^\mu_{\Sigma_1}$, respectively, we have

Proposition 1.1.

$$\int_{\Sigma_1} J^V_\mu(\phi) n^\mu_{\Sigma_1}\,\mathrm{dVol}_{\Sigma_1} + \int_B K^V(\phi)\,\mathrm{dVol} + \int_B FV^\nu\partial_\nu\phi\,\mathrm{dVol} = \int_{\Sigma_0} J^V_\mu(\phi) n^\mu_{\Sigma_0}\,\mathrm{dVol}_{\Sigma_0}.$$

$$\int_{\Sigma_1} J^{V,w}_\mu(\phi) n^\mu_{\Sigma_1}\,\mathrm{dVol}_{\Sigma_1} + \int_{\mathcal{B}} K^{V,w}(\phi)\,\mathrm{dVol} + \int_{\mathcal{B}} \left(\frac{1}{4}Fw\phi + FV^\nu\partial_\nu\phi\right)\mathrm{dVol} = \int_{\Sigma_0} J^{V,w}_\mu(\phi) n^\mu_{\Sigma_0}\,\mathrm{dVol}_{\Sigma_0}.$$

In particular, if we choose V Killing, vanishing source term $F = 0$, then we obtain a conservation law

$$\int_{\Sigma_1} J^V_\mu(\phi) n^\mu_{\Sigma_1}\,\mathrm{dVol}_{\Sigma_1} = \int_{\Sigma_0} J^V_\mu(\phi) n^\mu_{\Sigma_0}\,\mathrm{dVol}_{\Sigma_0}. \tag{1.3}$$

This is a manifestation of Noether's Theorem, which states that a differentiable one-parameter family of symmetries gives rise to a conservation law.

Remark 1.2. Though the preceding derivation only works for real-valued function ϕ, it is easy to be generalized to complex valued solutions. Suppose $\Box_g\phi = F$ with ϕ complex-valued, then we instead define

$$T_{\mu\nu} = \frac{1}{2}(\partial_\mu\phi\partial_\nu\bar{\phi} + \partial_\mu\bar{\phi}\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\bar{\phi} = \mathrm{Re}(\partial_\mu\phi\partial_\nu\bar{\phi}) - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\bar{\phi}$$

Then it satisfies

$$\nabla^\mu T_{\mu\nu} = \mathrm{Re}(\bar{F}\partial_\nu\phi).$$

Thus in the following propositions, the statement works for complex-valued solutions though we only prove it in the real case just for simplicity.

Proposition 1.3. *In Minkowski spacetime, for $\Box_m\phi = 0$, we have*

$$\int_{t=t_1}\left(|\partial_t\phi|^2 + \sum_{j=1}^3|\partial_j\phi|^2\right)dx_{t_1} = \int_{t=t_0}\left(|\partial_t\phi|^2 + \sum_{j=1}^3|\partial_j\phi|^2\right)dx_{t_0}$$

Proof. By choosing $V = \partial_t$ Killing, we notice that

$$J^V_0 = T_{00} = \frac{1}{2}\left((\partial_t\phi)^2 + \sum_{j=1}^3(\partial_j\phi)^2\right).$$

Then the proposition follows from (1.3) by choosing $\Sigma_0 = \{t = t_0\}$ and $\Sigma_1 = \{t = t_1\}$ with $t_0 < t_1$. ◻

Proposition 1.4. *In Schwarzschild spacetime, for $\Box_g\phi = 0$, we have*

$$\begin{aligned} E^{Schw}_{t_0}[\phi] :&= \int_{const\ t=t_0\ slice} \frac{1}{\sqrt{1-\mu}}\left(|\partial_t\phi|^2 + |\partial_{r^*}\phi|^2 + \frac{1-\mu}{r^2}|\nabla\!\!\!/\phi|^2\right)\mathrm{dVol}_t \\ &= \int_{const\ t=t_0\ slice}\left(|\partial_t\phi|^2 + |\partial_{r^*}\phi|^2 + \frac{1-\mu}{r^2}|\nabla\!\!\!/\phi|^2\right) r^2 dr^*\wedge d\sigma_{\mathbb{S}^2} \end{aligned}$$

is conserved, where $\mu = 2M/r$.

Proof. Recall that $V = T = \partial_t$ is Killing as well. In (t, r^*, θ) coordinates,

$$g = -(1-\mu)\, dt^2 + (1-\mu)\,(dr^*)^2 + r^2 d\sigma_{\mathbb{S}^2}$$

and hence

$$\begin{aligned} J^T_\mu(\phi) n_t^\mu &= \frac{1}{\sqrt{1-\mu}} T_{00}(\phi) = \frac{1}{\sqrt{1-\mu}} \left((\partial_t\phi)^2 + \frac{1-\mu}{2}\left(\frac{1}{1-\mu}(-(\partial_t\phi)^2 + (\partial_{r^*}\phi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\phi|^2 \right)\right) \\ &= \frac{1}{2\sqrt{1-\mu}} \left((\partial_t\phi)^2 + (\partial_{r^*}\phi)^2 + \frac{1-\mu}{r^2}|\nabla\!\!\!/\phi|^2 \right), \end{aligned}$$

where we choose $n_t = \frac{1}{\sqrt{1-\mu}}\partial_t$. Then

$$n_t^\flat = g_{\mu\nu} n_t^\nu\, dx^\mu = -\sqrt{1-\mu}\, dt \Rightarrow \mathrm{dVol}_t = r^2\sqrt{1-\mu}\, dr^* \wedge d\sigma_{\mathbb{S}^2}.$$

□

1.3. **Mode solutions.** Note that when $a = 0$, the Kerr metric reduces to the Schwarzschild metric, which is spherically symmetric. Thus, it is clear that the wave equation $\Box_{g_{0,M}}\phi = 0$ is separable. Nevertheless, the wave equation $\Box_{g_{a,M}}\phi = 0$ remains separable as first discovered by [Car68]. Take $g = g_{a,M}$, we record

$$\begin{aligned} \frac{e^{i\omega t}e^{-im\phi}}{\rho^2}\Box_g\left(e^{-i\omega t}e^{im\phi}\psi_0(r,\theta)\right) = \partial_r\left(\Delta\partial_r\psi_0\right) + \left(\frac{(r^2+a^2)^2\omega^2 - 4Mamr\omega + a^2m^2}{\Delta} - a^2\omega^2\right)\psi_0 \\ + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi_0\right) - \left(\frac{m^2}{\sin^2\theta} - a^2\omega^2\cos^2\theta\right)\psi_0. \end{aligned} \tag{1.4}$$

We remark that the separability follows from the presence on Kerr of a Killing tensor of order 2 [WP70]. See also [Wal84, p.444].

Then we call

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) - \left(\frac{m^2}{\sin^2\theta} - a^2\omega^2\cos^2\theta\right)S + \lambda S = 0 \tag{1.5}$$

the angular ODE for $S = S(\theta)$ and

$$\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + \left(\frac{(r^2+a^2)^2\omega^2 - 4Mamr\omega + a^2m^2}{\Delta} - a^2\omega^2\right)R - \lambda R = 0 \tag{1.6}$$

the radial ODE for $R = R(r)$.

For fixed ω and m, we consider the angular ODE (1.5). By defining $\widetilde{S}(\theta,\varphi) := e^{im\varphi}S(\theta)$, we notice that $\widetilde{S}$ satisfies

$$\Delta_{\mathbb{S}^2}\widetilde{S} = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\widetilde{S}\right) + \frac{1}{\sin\theta}\partial_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi\widetilde{S}\right) = -\lambda\widetilde{S} - a^2\omega^2\cos^2\theta\widetilde{S}.$$

In particular, when $a = 0$, this reduces to the eigenvalue problem of $\Delta_{\mathbb{S}^2}$. When $a \neq 0$, we still look at the original ODE (1.5) and then this can be viewed as a singular Sturm-Liouville problem with prescribed boundary $\widetilde{S}(\theta,\phi)$ is smooth on $\mathbb{S}^2$. By performing a change of variable $x = \cos\theta$, (1.5) is equivalent to

$$\partial_x((1-x^2)\partial_x S) - \frac{m^2}{1-x^2}S + a^2\omega^2 x^2 S = -\lambda S,$$

which is called the oblate spheroidal wave equation [ZD22, Look-Up Technique] and the references therein. The eigenfunctions are given by superpositions of associated Legendre functions. See also [DRS16, Section 5.2.1] and [DHR19, Section 6.2.1].

Remark 1.5. It seems to me that it's hard to show that the eigenfunctions span $L^2(\sin\theta\, d\theta)$ simply using the abstract Sturm-Liouville theory since it is a singular Sturm-Liouville problem. I feel it pretty hard to check limit point conditions at the endpoints $\theta = 0, \pi$. Instead, I believe it is easier to check the oblate spheroidal harmonics spans $L^2(\sin\theta\, d\theta)$.

Then the completeness of eigenfunctions for angular ODES allows us to pass from the nonexistence of mode solutions to the nonexistence of solutions of the form

$$e^{-i\omega t}e^{i\omega\phi}A(r,\theta)$$

for A satisfying suitable boundary conditions.

For the radial ODE (1.6), we consider its inhomogeneous version

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) - \left(-(r^2+a^2)^2\omega^2 + 4Mamr\omega - a^2m^2 + \Delta(a^2\omega^2+\lambda)\right)R = \Delta\hat{F}, \tag{1.7}$$

where $\hat{F}$ is a smooth compactly supported function on (r_+,∞). Using $\partial_r = \frac{r^2+a^2}{\Delta}\partial_{r^*}$ and we aim to enact a conjugation $u(r^*) = A(r)R(r)$ so that the equation for u does not have first order term. It is easy to notice that A is chosen to be

$$2(r^2+a^2)A' + (r^2+a^2)'A = 0.$$

Therefore, we work with

$$u(r^*) = (r^2+a^2)^{\frac{1}{2}}R(r),$$

which satisfies

$$\begin{aligned} u'' + (\omega^2 - V)u &= \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}}\hat{F} =: H(r^*), \\ V &:= \frac{4Mamr\omega - a^2m^2 + \Delta(\lambda + a^2\omega^2)}{(r^2+a^2)^2} + \frac{\Delta}{(r^2+a^2)^4}\left(a^2\Delta + 2Mr(r^2-a^2)\right). \end{aligned} \tag{1.8}$$

We remark that V is real in the Schwarzschild case ($a = 0$).

To facilitate the analysis near $r = r_+$, we introduce a further change of variable

$$\omega_0 := \omega - \frac{am}{2Mr_+}, \quad V_0 := V + \omega_0^2 - \omega^2.$$

Then $V_0 = O(r - r_+)$ and u still satisfies an equation with a similar form

$$u'' + (\omega_0^2 - V_0)u = \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}}\hat{F}. \tag{1.9}$$

The definition of ω_0 is motivated by the superradiant frequency consideration.

Definition 1.6. Let $(\mathcal{M}, g = g_{a,M})$ be a sub-extremal Kerr spacetime. A smooth solution ϕ to the wave equation

$$\Box_g \phi = 0$$

is called a mode solution if there exist parameters $(\omega, m, l) \in \mathbb{C} \setminus \{0\} \times \mathbb{Z} \times \mathbb{Z}_{\geq m}$ such that

$$\phi(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\varphi} S_{\omega,m,l}(\theta) R_{\omega,m,l}(r), \tag{1.10}$$

where we might often omit the arguments ω, m, l. Here,

(1) S satisfies the angular ODE (1.5) (with eigenvalue λ) associated with the boundary condition that $\widetilde{S}(\theta, \varphi)$ extends smoothly on $\mathbb{S}^2$.
(2) R satisfies the radial ODE (1.6) with boundary conditions

$$R \sim (r - r_+)^{\frac{i(am-2Mr_+\omega)}{r_+-r_-}} \text{ at } r = r_+, \qquad R \sim \frac{e^{i\omega r^*}}{r} \text{ at } r = \infty, \tag{1.11}$$

at the horizon and infinity, respectively.

Remark 1.7. The boundary conditions in (1.11) for R can be translated into the corresponding version for u (see (1.8)) as follows :

$$u \sim (2Mr_+)^{\frac{1}{2}} (r - r_+)^{\frac{i(am-2Mr_+\omega)}{r_+-r_-}} \text{ at } r^* = -\infty, \qquad u \sim e^{i\omega r^*} \text{ at } r^* = \infty, \tag{1.12}$$

where we use $r_+^2 - 2Mr_+ + a^2 = \Delta|_{r=r_+} = 0$.

Remark 1.8. The boundary conditions required here will become clear in a moment, after we notice $r = r_+$ is a regular singular point and $r = +\infty$ is an irregular singular point. The notation $\sim$ in the boundary condition (1.11) at $r = r_+$ means that R has a power series expansion in $(r - r_+)$ with a prefactor $(r - r_+)^{\frac{i(am-2Mr_+\omega)}{r_+-r_-}}$. Similarly, the boundary condition (1.11) at the irregular singular point $r = +\infty$ means that R a power series expansion in r^{-1} with a prefactor $r^{-1} e^{i\omega r^*}$.

Now we derive the boundary conditions (1.11). Recall that $\Delta = (r - r_+)(r - r_-)$, by rewriting (1.6) into

$$\frac{d^2R}{dr^2} + \left(\frac{1}{r - r_+} + \frac{1}{r - r_-} \right) \frac{dR}{dr} + \left(\frac{(r^2 + a^2)^2\omega^2 - 4Mamr\omega + a^2m^2}{(r - r_+)^2(r - r_-)^2} - \frac{a^2\omega^2 + \lambda}{(r - r_+)(r - r_-)} \right) R = 0.$$

Then it is easy to see that $r = r_\pm$ are regular singular points. At $r = r_+$, the indicial equation (see [Olv74, Section 5.4]) is given by

$$\alpha(\alpha - 1) + \alpha + \frac{(r_+^2 + a^2)^2\omega^2 - 4Mamr_+\omega + a^2m^2}{(r_+ - r_-)^2} = 0 \Rightarrow \alpha^2 = -\frac{(am - 2Mr_+\omega)^2}{(r_+ - r_-)^2},$$

where we use the fact $r = r_+$ solves $\Delta = r^2 - 2Mr + a^2 = 0$ in the last step. Since we would expect the solution takes the form

$$R = (r - r_+)^\alpha \sum_{j=0} c_j (r - r_+)^j$$

and have finite energy along constant t slices when $\operatorname{Im}\omega > 0$ (one can see the numerology from Proposition 1.4), we choose

$$\alpha = i\frac{am - 2Mr_+\omega}{r_+ - r_-}.$$

Alternatively, it is also easy to just think as follows. We want the mode solution with favorable boundary conditions to at least not blow up at $r = r_+$, this amounts to ruling out the choice $-i\frac{am-2Mr_+\omega}{r_+-r_-}$.

On the other hand, to discuss the boundary condition of R near $r = \infty$, it is easy to investigate the ODE satisfied by u instead. From (1.8), one knows that the characteristic equation for the normal series expansion (see [Olv74, Section 7.1]) in $1/r^*$ is given by

$$u(r^*) = e^{\lambda r^*}(r^*)^{\mu} \sum_{j=0}^{\infty} \frac{c_s}{(r^*)^j}, \quad \lambda^2 + \omega^2 = 0, \quad 2\lambda\mu = 0.$$

To ensure that the energy on constant t slices to be finite when $\operatorname{Im}\omega > 0$, we choose $\lambda = i\omega$ and $\mu = 0$, which is in line with the boundary condition for R (1.11). (Choosing $\lambda = +i\omega$ instead of $-i\omega$ makes u to be exponentially decaying at ∞ instead of growing, which is the favorable boundary condition we would like.) In fact, we will see that the energy on asymptotically flat hypersurfaces is finite when $\operatorname{Im}\omega > 0$ while the energy along hyperboloidal hypersurfaces is finite when $\operatorname{Im}\omega \leq 0$. See [Shl15, Appendix D].

Remark 1.9. Note that for real mode solutions, we have oscillatory behavior near $r = +\infty$. For the work [Shl14] on the superradiant instability on Klein Gordon equation on Kerr, the idea of microlocal energy current one will see later is still applied. However, due to the structure of Klein-Gordon equation, $\Box_g\phi = \mu^2\phi$, the boundary condition for real mode at $+\infty$ is still decaying thanks to the same kind of ODE analysis as above. Specifically, $R \sim e^{i\sqrt{\omega^2-\mu^2}r^*}/r$.

Except the idea microlocal energy current, [Shl14] applies a variational argument by considering the functional whose Euler-Lagrange equation is the radial ODE for u. Then one can construct an eigenvector for the radial ODE and the existence of a mode solution follows by tweaking m.

After the discussion about the derivation of boundary conditions, it is then natural to introduce the following definitions.

Definition 1.10. Let the parameters $|a| < M$ be fixed. Then define $u_{\text{hor}}(r^*, \omega, m, l)$ to be the unique function satisfying

(1) $u''_{\text{hor}} + (\omega^2 - V)\, u_{\text{hor}} = 0$.
(2) $u_{\text{hor}} \sim (r - r_+)^{\frac{i(am-2Mr_+\omega)}{r_+-r_-}}$ near $r^* = -\infty$.
(3) $\left| \left((r(r^*) - r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}} u_{\text{hor}} \right)(-\infty) \right|^2 = 1$.

Definition 1.11. Let the parameters $|a| < M$ be fixed. Then define $u_{\text{out}}(r^*, \omega, m, l)$ to be the unique function satisfying

(1) $u''_{\text{out}} + (\omega^2 - V)\, u_{\text{out}} = 0$.

(2) $u_{\text{out}} \sim e^{i\omega r^*}$ near $r^* = \infty$.
(3) $\left|\left(e^{-i\omega r^*} u_{\text{out}}\right)(\infty)\right|^2 = 1$.

Definition 1.12. We define the Wronskian as

$$W(\omega, m, l) := u'_{\text{out}}(r^*) u_{\text{hor}}(r^*) - u'_{\text{hor}}(r^*) u_{\text{out}}(r^*),$$

which is independent of r^*.

Remark 1.13. The Wronskian will vanish if and only if u_{out} and u_{hor} are linearly dependent, i.e. there exists a non-trivial solution to (1.8) with $H = 0 \Leftrightarrow W = 0 \Leftrightarrow |W^{-1}| = \infty$. "Quantitative mode stability" consists of producing an upper bound for $|W^{-1}|$ with an explicit dependence on a, M, ω, m, and l.

1.4. **A unique continuation lemma.**

Lemma 1.14. *Suppose that we have a solution $u(r^*) : (-\infty, \infty) \to \mathbb{C}$ to the ODE*

$$u'' + \left(\omega^2 - V\right) u = 0$$

such that
(1) $\omega \in \mathbb{R} \setminus \{0\}$,
(2) $u \in L^\infty$ and $\left(|u'|^2 + |u|^2\right)(\infty) = 0$,
(3) V is real, $V \in L^\infty$, $V = O\left(r^{-1}\right)$ as $r \to \infty$, and $V' = O\left(r^{-2}\right)$ as $r \to \infty$.
Then u is identically 0.

Proof. We define the virial current

$$Q^y := y|u'|^2 + y(\omega^2 - V)|u|^2, \tag{1.13}$$

where $y(r^*)$ is a suitably chosen function

$$y(r^*) := \exp(-B \int_{r^*}^{\infty} \zeta(r')\, dr'). \tag{1.14}$$

Here, B is a large constant to be determined and ζ is a fixed positive function on $(-\infty, \infty)$ such that

$$\zeta(r') = \begin{cases} 1 & \text{near } r' = -\infty \\ r^{-2} & \text{near } r' = \infty \end{cases}.$$

With these requirements being satisfied, we have $y(\infty) = 1$ and $y(-\infty) = 0$. Note that $Q^y(\pm\infty) = 0$ and

$$(Q^y)'(r^*) = y'|u'|^2 + y'\omega^2|u|^2 - (yV)'|u|^2,$$

we know that

$$\int_{-\infty}^{\infty} y'|u'|^2 + y'\omega^2|u|^2\, dr^* = \int_{-\infty}^{\infty} (yV)'|u|^2\, dr^* \tag{1.15}$$

Set $\chi(r^*)$ to be a function identically 1 on $(-\infty, R]$ and 0 on $[R+1, \infty)$, where R is to be determined. We then write

$$V = V_1 + V_2 := \chi V + (1 - \chi)V.$$

Therefore, we could integrate by parts and estimate

$$\left|\int_{-\infty}^{\infty}(yV_1)'|u|^2\,dr^*\right| = 2\left|\int_{-\infty}^{\infty}yV_1\mathrm{Re}(\bar{u}u')\,dr^*\right| \leq \varepsilon\int_{-\infty}^{\infty}y'|u'|^2\,dr^* + \varepsilon^{-1}\int_{-\infty}^{\infty}\left(\frac{yV_1}{y'\omega}\right)^2 y'\omega^2|u'|^2$$
$$\leq \varepsilon\int_{-\infty}^{\infty}y'|u'|^2\,dr^* + \varepsilon^{-1}\left(\frac{\|V\|_{L^\infty}}{\omega B(\inf_{-\infty<r'<R+1}\zeta(r'))}\right)^2\int_{-\infty}^{\infty}y'\omega^2|u'|^2\,dr^*.$$

On the other hand,

$$\left|\int_{-\infty}^{\infty}(yV_2)'|u|^2\,dr^*\right| \leq \left|\int_{-\infty}^{\infty}y'V_2|u|^2\,dr^*\right| + \left|\int_{-\infty}^{\infty}yV_2'|u|^2\,dr^*\right|$$
$$\leq \left(\frac{\|V_2\|_{L^\infty}}{\omega^2} + \omega^{-2}\|\frac{V_2'}{\zeta}\|_{L^\infty}\right)\int_{-\infty}^{\infty}y'\omega^2|u'|^2\,dr^*,$$

where according to the third assumption, V_2'/ζ is in L^∞ and $\|V_2\|_{L^\infty}$ can be taken small by choosing R large. Therefore, taking ε small enough and then taking B, R large enough, we could make the right hand side of (1.15) smaller than an arbitrary small multiple of the left hand side, which is a contradiction due to the coercivity of the left hand side unless

$$\int_{-\infty}^{\infty}y'|u'|^2 + y'\omega^2|u|^2\,dr^* = 0.$$

In particular, $u \equiv 0$. □

Remark 1.15. In the third assumption, the notation $V = O(r^{-1})$ simply means $|V| \lesssim r^{-1}$ for r sufficiently large. This does not require the asymptotes $V \sim Cr^{-1}$.

1.5. **Teukolsky equation.** Now we discuss a general version of the linearization of the wave equation with Kerr background, the Teukolsky equation. In Boyer-Lindquist coordinates, the Teukolsky equation [Teu73]

$$\begin{aligned}\Box_g\alpha^{[s]} + \frac{2s}{\rho^2}(r-M)\partial_r\alpha^{[s]} + \frac{2s}{\rho^2}\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\partial_\phi\alpha^{[s]} \\ +\frac{2s}{\rho^2}\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t\alpha^{[s]} + \frac{1}{\rho^2}\left(s - s^2\cot^2\theta\right)\alpha^{[s]} = 0\end{aligned} \tag{1.16}$$

can be considered for arbitrary values of $s \in \frac{1}{2}\mathbb{Z}$. For $s = \pm 2$, it plays a central role, as it describes the dynamics of the extremal curvature components of the metric in the Newman–Penrose formalism [NP62], where it is related to the linearzation of Einstein equation around the Kerr solutions. For $s = 0$, the Teukolsky equation (1.16) reduces to the wave equation (1.4). For $s = \pm 1$, it describes the evolution of the extreme components of the Maxwell equations in a null frame.

As Teukolsky noted in his seminal paper [Teu73], by analogy with the wave equation case [Car68], the Teukolsky equation (1.16) is separable, i.e. it admits separable solutions:

$$\alpha^{[s]}(t,r,\theta,\phi) = e^{-i\omega t}e^{im\phi}S^{[s],\,a\omega}_{m,\lambda}(\theta)\Delta^{-\frac{s+1}{2}}R^{[s],\,a,\omega}_{m,\lambda}(r)\,, \tag{1.17}$$

for $\omega \in \mathbb{C}$, $m - s \in \mathbb{Z}$ and a separation constant. Plugging (1.17) into (1.16), we find that $S^{[s],\,a\omega}_{m,\lambda}$ and $R^{[s],\,a,\omega}_{m,\lambda}$ each satisfy ODEs given below.

The angular ODE verified by $S^{[s],\nu}_{m,\lambdabar}$ is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}S^{[s],\nu}_{m,\lambdabar}(\theta)\right)-\left(\frac{(m+s\cos\theta)^2}{\sin^2\theta}-\nu^2\cos^2\theta+2\nu s\cos\theta\right)S^{[s],\nu}_{m,\lambdabar}(\theta)+\lambdabar S^{[s],\nu}_{m,\lambdabar}(\theta)=0\,, \tag{1.18}$$

where we replace $a\omega$ by a parameter $\nu\in\mathbb{C}$.

Fix $M>0$. For $|a|\le M$, $s\in\frac{1}{2}\mathbb{Z}$, $m-s\in\mathbb{Z}$, $\omega\in\mathbb{C}$ and $\lambdabar\in\mathbb{C}$, the radial ODE verified by $R^{[s],a,\omega}_{m,\lambdabar}(r)$ in (1.17) is, for $r\in(r_+,\infty)$,

$$\begin{aligned}\Delta\frac{d^2}{dr^2}R^{[s],a,\omega}_{m,\lambdabar}(r)+\frac{[\omega(r^2+a^2)-am-is(r-M)]^2}{\Delta}R^{[s],a,\omega}_{m,\lambdabar}(r)\\+\left(\frac{M^2-a^2}{\Delta}+4is\omega r-\lambdabar-a^2\omega^2+2am\omega\right)R^{[s],a,\omega}_{m,\lambdabar}(r)=0\,.\end{aligned} \tag{1.19}$$

Like before, we focus on studying (1.19) under suitable boundary conditions (see [CT22, Definition 2.3]).

2. Mode stability

2.1. **Physical space method for the mode stability.** First, it is to deduce from the conservation law Proposition 1.4 to rule out the existence of mode solutions with $\operatorname{Im}\omega>0$ (exponential growing modes). Suppose not, by putting

$$\phi(t,r,\theta,\varphi)=e^{-i\omega t}e^{im\varphi}S(\theta)R(r):=e^{-i\omega t}\psi(r,\theta,\phi),\quad \omega=\omega_R+i\omega_I,$$

we arrive at

$$|\partial_t\phi|^2=|\omega|^2e^{2\omega_I t}|\psi|^2,\quad |\partial_{r^*}\phi|^2=e^{2\omega_I t}|\partial_{r^*}\psi|^2,\quad |\nabla\!\!\!\!/\,\phi|^2=e^{2\omega_I t}|\nabla\!\!\!\!/\,\psi|^2,$$

which contradicts the energy estimates $E^{Schw}_t[\phi]\le E^{Schw}_0[\phi]$ for t sufficiently large. Note that the finiteness of initial energy is due to the boundary conditions for $S(\theta)$ and $R(r)$.

However, for real modes, we do not have the exponential growing $e^{2\omega_I t}$ anymore to derive a contradiction. Then we need to consider the energy estimates with part of the null infinity $\mathcal{I}^+$ involved.

2.2. **Mode stability of Schwarzschild.** We consider the analogue of energy flux from a microlocal point of view. We define the microlocal energy current :

$$Q_T(r^*):=\operatorname{Im}(u'\overline{\omega u}),$$

where u solves (1.8) with $\hat{F}=0$. Let us show how the microlocal energy can be used to give a short proof of mode stability.

First, we falsify the existence of a growing mode solution. Suppose we have a mode solution with corresponding $u(r^*)$ and $\omega=\omega_R+i\omega_I$ for some $\omega_I>0$.

When $a=0$, $\lambda=l(l+1)$ and $V=r^{-4}(r-2M)(rl(l+1)+2M)$. We compute

$$-Q_T'=\omega_I|u'|^2+\operatorname{Im}((\omega^2-V)\overline{\omega})|u|^2=\omega_I\left(|u'|^2+\big(|\omega|^2+\frac{(r-2M)(rl(l+1)+2M)}{r^4}\big)|u|^2\right). \tag{2.1}$$

Thanks to the boundary condition (1.12), Q_T vanishes at $r^* = \pm\infty$ and hence $u \equiv 0$ since $-Q_T'$ is positive definite with respect to (u', u).

Next, we show that there are no real modes. Suppose by contradiction that we have a mode solution with corresponding $u(r^*)$ and $\omega \in \mathbb{R} \setminus \{0\}$. From (2.1), we know that

$$Q_T(\infty) = Q_T(-\infty).$$

By writing out the explicit computation, we arrive at

$$\omega^2 |u(\infty)|^2 + 2Mr_+\omega^2 |u(-\infty)|^2 = 0. \tag{2.2}$$

Moreover, since u is smooth by the definition of a mode solution, we know that $u \in L^\infty$ according to the boundary value in (1.12). Then it follows from the unique continuation result Lemma 1.14 that u is zero everywhere.

Remark 2.1. The motivation of the microlocal energy follows from the frequency view of the standard energy method. Recall that by multiplying the equation by $\overline{\partial_t u}$, we could use integration by parts to derive the standard energy estimates. On the Fourier side, $\overline{\partial_t u}$ corresponds to $i\bar{\omega}\bar{\hat{u}}$. This somehow suggests that we could consider the differential identity

$$0 = i\bar{\omega}\bar{u}(u'' + (\omega^2 - V)u) = -i\bar{\omega}|u'|^2 + i(\bar{\omega}\bar{u}u')' + i\omega|\omega|^2|u|^2 - i\bar{\omega}V|u|^2.$$

The second term in the last expression explains the choice of Q_T'. The consideration above is a combination of physical side and Fourier side, i.e. in phase space, motivating the name microlocal energy.

2.3. **Mode stability of subextremal Kerr outside superradiant frequencies.** The superradiant frequencies were first discussed in [Zel71], which is a phase space manifestation of the fact the Killing vector field ∂_t fails to be timelike in the ergoregion $\Delta - a^2\sin^2\theta < 0$. In other regime, we may take advantage of this timelike Killing vector field to derive conservation laws as in Section 1.2. However, when this Killing vector field is not timelike, the associated physical space energy flux may be negative. See [DR12, Section 5.1, 7.1.1]. At the level of mode analysis for $\omega \in \mathbb{R}$, this difficulty arises in the range

$$0 \le \omega < \frac{am}{2Mr_+}.$$

This range can be read off from repeating the argument in the preceding subsection for Schwarzschild. If one defines Q_T in the same way and compute Q_T' for a real mode u. Then $Q_T(\infty) = Q_T(-\infty)$ yields

$$\omega^2|u(\infty)|^2 - \omega(am - 2Mr_+\omega)|u(-\infty)|^2 = 0. \tag{2.3}$$

When $\omega(am - 2Mr_+\omega) > 0$, this fails to give an estimate where we could apply the unique continuation result, which is exactly the superradiant frequency range.

Remark 2.2. In general, the existence of superradiant frequency may cause the existence of growing modes. Remarkably, in [Whi89], Whiting found integral and differential transformations so that we could reduce to a case with a new metric without ergoregion. Then standard physical space method applies to conclude the mode stability. However, in this note, we adopt the method used [Shl15], where the exposition avoids introducing the differential transform and a new metric (see [Whi89, Section IV, VI]). This is due to the observation related to the angular ODE mentioned in Remark 2.8 so that we avoid the need to make a transformation on the angular ODE as well.

2.4. **Mode stability of subextremal Kerr in full generality.** As we mentioned above, we would introduce the so-called Whiting's integral transformation

$$\tilde{u}(x^*) := (x^2+a^2)^{\frac{1}{2}} (x-r_+)^{-2iM\omega} e^{-i\omega x} \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)} (r-r_-)^{\eta} (r-r_+)^{\xi} e^{-i\omega r} R(r)\, dr, \tag{2.4}$$

where

$$\eta := \frac{-i(am-2Mr_-\omega)}{r_+-r_-}, \quad \xi := \frac{i(am-2Mr_+\omega)}{r_+-r_-}.$$

Note that the integral should be understood in the sense that it is defined for x replaced by z in the upper half plane and then $y=\operatorname{Im} z$ is taken $y \to 0_+$.

Remark 2.3. One may notice that the transformation looks like the Fourier transform. This fact shall be used in the proof of Theorem 2.7 and Theorem 2.9 to derive the injectivity of the map $R \mapsto \tilde{u}$.

Now we state the main properties related to the integral transformation.

Proposition 2.4. *Let $\operatorname{Im}\omega \geq 0$, $\omega \neq 0$, R solve the inhomogeneous radial ODE (1.7), and R satisfy the boundary conditions (1.11). Define $\tilde{u}$ via Whiting's integral transformation (2.4). Then $\tilde{u}(x)$ is C^∞ on (r_+,∞) and, letting primes denote x^*-derivatives, satisfies*

$$\tilde{u}'' + \Phi\tilde{u} = \tilde{H},$$

where

$$\begin{aligned}
\tilde{H}(x^*) &:= \frac{(x-r_+)(x-r_-)}{(x^2+a^2)^2}\tilde{G}(x), \quad \Phi(x^*) := \frac{(x-r_-)\tilde{\Phi}_1(x)}{(x^2+a^2)^2} - \tilde{\Phi}_2(x),\\
\tilde{G}(x) &:= (x^2+a^2)^{1/2}(x-r_+)^{-2iM\omega}e^{-i\omega x}\int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}\hat{F}(r)\, dr,\\
\tilde{\Phi}_1(x) &:= \omega^2(x-r_+)^2(x-r_-) - \left(4M\omega^2 + \frac{4\omega(am-2Mr_+\omega)}{r_+-r_-}\right)(x-r_-)(x-r_+)\\
&\qquad + 4M^2\omega^2(x-r_-) + \left(2am\omega - \lambda_{\omega ml} - a^2\omega^2\right)(x-r_+),\\
\tilde{\Phi}_2(x) &:= \frac{(x-r_+)(x-r_-)}{(x^2+a^2)^4}\left(a^2(x-r_+)(x-r_-) + 2Mx(x^2-a^2)\right).
\end{aligned} \tag{2.5}$$

In particular, we remark that it is only important to know that Φ is real for proving the mode stability result on the real axis.

The following proposition translates the boundary conditions of u to $\tilde{u}$. Though the following is a crude version, it is already sufficient to rule out growing modes.

Proposition 2.5. *If $\operatorname{Im}\omega > 0$, then*

(1) $\tilde{u} = O\left((x-r_+)^{2M\operatorname{Im}\omega}\right)$ as $x \to r_+$.

(2) $\tilde{u}' = O\left((x-r_+)^{2M\mathrm{Im}\,\omega}\right)$ *as* $x \to r_+$.
(3) $\tilde{u} = O\left(e^{-x\mathrm{Im}\,\omega}x^{1+2M\mathrm{Im}\,\omega}\right)$ *as* $x \to \infty$.
(4) $\tilde{u}' = O\left(e^{-x\mathrm{Im}\,\omega}x^{1+2M\mathrm{Im}\,\omega}\right)$ *as* $x \to \infty$.

For the analysis of real modes, we need a more precise version.

Proposition 2.6. *If* $\omega \in \mathbb{R}\setminus\{0\}$, *then*
(1) $\tilde{u}$ *is uniformly bounded.*
(2) $|\tilde{u}(\infty)|^2 = \frac{(r_+-r_-)^2|\Gamma(2\xi+1)|^2}{8M\omega^2 r_+}|u(-\infty)|^2$.
(3) $\tilde{u}'$ *is uniformly bounded.*
(4) $\tilde{u}' - i\omega\tilde{u} = O(x^{-1})$ *at* $x^* = \infty$.
(5) $\tilde{u}' + \frac{i\omega(r_+-r_-)}{r_+}\tilde{u} = O(x-r_+)$ *at* $x^* = -\infty$.

We defer the proof of these properties. Instead, we use these to conclude the proof of mode stability for Kerr.

Theorem 2.7 (Whiting [Whi89])**.** *There exist no non-trivial mode solutions corresponding to* $\mathrm{Im}\,\omega > 0$ *for subextremal Kerr background* ($|a| < M$)*.*

Proof. Following the same computation in (2.1) for $\tilde{u}$, we obtain

$$-\tilde{Q}_T' = \omega_I|\tilde{u}'|^2 + \mathrm{Im}\,(\Phi\bar{\omega})|\tilde{u}|^2.$$

An involved computation shows that $\mathrm{Im}\,(\Phi\bar{\omega}) \geq 0$ and hence $\tilde{u} \equiv 0$. In terms of R, this implies that

$$\tilde{R}(x) := \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\,dr,$$

vanishes for $x \in (r_+, \infty)$.

To see that this implies that R vanishes, we first extend R by 0 to all of R and note that the Fourier transform of $(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)$ is, up to a change of variables,

$$\widehat{R}(z) := \int_{-\infty}^{\infty} e^{2i|\omega|^2 z(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\,dr.$$

In view of the support of R, we know that $\widehat{R}(z)$ is holomorphic in the upper half plane. Then the vanishing of R follows from the vanishing of $\widehat{R}$ along $\{\frac{y}{\bar{\omega}} : y \in (1,\infty)\}$. □

Remark 2.8. It is worthwhile to remark that $\mathrm{Im}\,(\Phi\bar{\omega}) \geq 0$ is not trivial. A direct computation of $\mathrm{Im}\,(\Phi\bar{\omega})$ gives obvious positive terms except for one with coefficients $-\mathrm{Im}\,((\lambda_{\omega ml} + a^2\omega^2)\bar{\omega})$. This turns out to be positive and one can see this by multiplying the angular ODE (1.5) by $\overline{\omega S_{\omega ml}}\sin\theta$, integration by parts and taking the imaginary part. Indeed,

$$\omega_I \int_0^{\pi}\left(\left|\frac{dS_{\omega ml}}{d\theta}\right|^2 + \left(\frac{m^2}{\sin^2\theta} + a^2|\omega|^2\sin^2\theta\right)|S_{\omega ml}|^2\right)\sin\theta d\theta$$
$$= -\int_0^{\pi}\left(\mathrm{Im}\left(\left(\lambda_{\omega ml} + a^2\omega^2\right)\bar{\omega}\right)\right)|S_{\omega ml}|^2\sin\theta d\theta \geq 0$$

Theorem 2.9 (Shlapentokh-Rothman [Shl15]). *There exist no non-trivial mode solutions corresponding to $\omega \in \mathbb{R} \setminus \{0\}$ for subextremal Kerr background ($|a| < M$).*

Proof. Using the last two properties in Proposition 2.6, we compute

$$\tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) = \frac{1}{2}\left(\omega^2|\tilde{u}(\infty)|^2 + |\tilde{u}'(\infty)|^2 + \omega^2\frac{r_+ - r_-}{r_+}|\tilde{u}(-\infty)|^2 + \frac{r_+}{r_+ - r_-}|\tilde{u}'(-\infty)|^2\right). \tag{2.6}$$

Then we notice that $\tilde{Q}_T' = 0$ and it follows from the same reasoning for (2.2)that the right hand side of (2.6) vanishes. To pass the vanishing of $\tilde{R}$ to the vanishing of R on the real line, we use the following fact :

The Fourier transform of a non-trivial function supported in $(0, \infty)$ cannot vanish on an open set (of the real line).

This then follows from extending $\tilde{R}$ to the upper half plane holomorphically and applying the Schwarz reflection principle to obtain an holomorphic extension on the complex plane. Then the vanishing on an open set of the real line implies the vanishing of R. □

2.5. Quantitative mode stability of subextremal Kerr for real mode parameters.

Theorem 2.10. *Let*

$$\mathscr{A} \subset \left\{(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}\right\}$$

be a set of frequency parameters with

$$C_{\mathscr{A}} := \sup_{(\omega,m,l)\in\mathscr{A}} \left(|\omega| + |\omega|^{-1} + |m| + |l|\right) < \infty.$$

Then

$$\sup_{(\omega,m,l)\in\mathscr{A}} \left|W^{-1}\right| \leq G(C_{\mathscr{A}}, a, M)$$

where the function G can, in principle, be given explicitly.

The proof consists of the following ingredients.

Proposition 2.11. *Consider $(\omega, m, l) \in \mathscr{A}$, u solving (1.8) with a smooth compactly supported inhomogeneity H. The following estimates hold*

$$|u(-\infty)|^2 \lesssim_{\mathscr{A}} (4\varepsilon)^{-1} \int_{r_+}^{\infty} |\hat{F}(r)|^2 \, dr + \varepsilon \int_{r_+}^{\infty} |R(r)|^2 \, dr \tag{2.7}$$

for $\varepsilon > 0$.

Proof. Combining

$$\tilde{Q}_T' = \omega \mathrm{Im}\,(\tilde{H}\bar{\tilde{u}})$$

with (2.6) and the second property in Proposition 2.6, we arrive at

$$|u(-\infty)|^2 \simeq |\tilde{u}(\infty)|^2 \lesssim \int_{-\infty}^{\infty} \mathrm{Im}\,(\tilde{H}\bar{\tilde{u}}) \, dr^*.$$

Moreover, it follows from Plancherel formula that

$$\int_{r_+}^{\infty} |x|^{-2}|\tilde{u}(x)|^2 \, dx \simeq \int_{r_+}^{\infty} |R(r)|^2 \, dr, \quad \int_{r_+}^{\infty} |x|^2|\tilde{H}(x)|^2 \, dx \simeq \int_{r_+}^{\infty} |\hat{F}(r)|^2 \, dr,$$

which concludes the result by Cauchy-Schwarz inequality. □

Now we would like to get rid of the term involving R on the right hand side of the estimate above.

Notice that $|u(-\infty)| \simeq |Q_T(-\infty)|^2$, we seek an microlocal energy based proof. In fact, we consider the virial current (1.13) and a similar computation reveals that

$$\int_{-\infty}^{\infty} y'|u'|^2 + y'\omega^2|u|^2\,dr^* \lesssim |Q_T(\infty)| + \int_{-\infty}^{\infty} y|H|^2r^2\,dr^*, \tag{2.8}$$

where y is defined as in (1.14). Notice that y, y' has exponential decay at negative infinity so this only gives a sufficient strong estimate away from the horizon. On the other hand, we could modify the definition of y to make it exponentially decay at $+\infty$ and rewrite

$$Q^y = y|u'|^2 + y(\omega_0^2 - V_0)|u|^2,$$

where ω_0 and V_0 are both defined in (1.9). Then a similar computation ends up with

$$\int_{-\infty}^{\infty} (-y'|u'|^2 - y'\omega_0^2|u|^2)\,dr^* \lesssim |\frac{\omega_0}{\omega}Q_T(-\infty)| + \int_{-\infty}^{\infty} y|H|^2r^2\,dr^*, \tag{2.9}$$

where the factor $\frac{\omega_0}{\omega}$ comes from the identities $Q^y(\infty) = 0$ and

$$Q_T(-\infty) = -\omega(am - 2Mr_+\omega)|u(-\infty)|^2 = 2Mr_+\omega\omega_0|u(-\infty)|^2 = 2Mr_+\frac{\omega}{2\omega_0}Q^y(-\infty),$$

where we use the computation result in (2.3) and the fact $V_0(-\infty) = 0$. Due to the presence of this factor, (2.9) gives an estimate near horizon when $|\omega_0|$ is away from 0.

We next deduce an estimate suitable for small ω_0, which is the so-called microlocal red-shift estimate.

Remark 2.12. I do not have a good idea why this is named as a red-shift estimate.

We consider the microlocal redshift current

$$Q^z_{\text{red}} := z\,|u' + i\omega_0 u|^2 - zV_0|u|^2 = Q^z + 2z\frac{\omega_0}{\omega}Q_T.$$

From the boundary condition (1.12), we know that $(u' + i\omega_0 u)(-\infty) = 0$, which allows z to blow up near horizon so that Q^z is still well-defined. In fact, we define

$$z(r^*) = -\frac{\zeta(r(r^*))}{V_0},$$

where ζ is a bump function identically 1 on $[r_+, r_+ + \varepsilon]$, 0 on $[r_+ + 2\varepsilon, \infty)$. One can verify that

$$(Q^z_{\text{red}})' \begin{cases} = z'|u' + i\omega_0 u|^2 + 2z\text{Re}((u' + i\omega_0 u)\bar{H}), & r \in [r_+, r_+ + \varepsilon], (z' \sim (r - r_+)^{-1} \text{ near horizon}) \\ \lesssim |u'|^2 + |u|^2 + |2z\text{Re}((u' + i\omega_0 u)\bar{H})|, & r \in [r_+ + \varepsilon, r_+ + 2\varepsilon] \text{ (treated as an error)} \\ = 0, & r \in [r_+ + 2\varepsilon, \infty) \end{cases},$$

where all prime notations denote d/dr^*. Moreover, $-|u(-\infty)|^2 = Q^z_{\text{red}}|^{\infty}_{-\infty}$ has a correct sign. Hence,

$$\int_{r_+}^{r_+ + \varepsilon} (r - r_+)^{-2}|u' + i\omega_0 u|^2\,dr \lesssim \int_{r_+ + \varepsilon}^{r_+ + 2\varepsilon} (|u'|^2 + |u|^2)\,dr + \int_{-\infty}^{\infty} |z\text{Re}((u' + i\omega_0 u)\bar{H})|\,dr^*,$$

where the factor $(r-r_+)^{-2}$ comes from $z' \sim (r-r_+)^{-1}$ in the corresponding range and the change of variable dr^* to dr.

For $h = \chi u$, where χ is a further cutoff to the interval $[r_+, r_+ + \varepsilon/2]$, we use integration by parts to write

$$\int_{r_+}^{\infty} |h|^2\, dr = \int_{r_+}^{\infty} |h|^2 \frac{d}{dr}(r-r_+)\, dr = -\int_{r_+}^{\infty} \frac{r^2+a^2}{r-r_-}\left((h'+i\omega_0 h)\bar{h} + (\overline{h'+i\omega_0 h})h\right)$$

and then using Cauchy-Schwarz to perform an estimate. Finally, we conclude

$$\int_{r_+}^{r_++\varepsilon} (r-r_+)^{-2}|u'+i\omega_0 u|^2\, dr + \int_{r_+}^{r_++\varepsilon} |u|^2\, dr \lesssim \int_{r_++\varepsilon}^{r_++2\varepsilon} (|u'|^2+|u|^2)\, dr + \int_{-\infty}^{\infty} |H|^2 r^2\, dr^*, \tag{2.10}$$

where the first term on the right hand side can be further estimated by the virial estimate (2.8) since it's away from the horizon.

Now we combine the three estimates (2.8), (2.9) and (2.10), suitable in different regimes respectively, we obtain

$$\int_{r_+}^{\infty} |R|^2\, dr \lesssim |Q_T(-\infty)| + |Q_T(\infty)| + \int_{r_+}^{\infty} |\hat{F}|^2\, dr.$$

For $|Q_T(\infty)|$, we estimate it using

$$Q_T' = \omega \mathrm{Im}\,(u\bar{H}),$$

that is,

$$|Q_T(\infty)| \le |Q_T(-\infty)| + \int |\omega \mathrm{Im}\,(u\bar{H})|\, dr \lesssim |Q_T(-\infty)| + \varepsilon^{-1}\int |\hat{F}|^2\, dr + \varepsilon \int |R|^2\, dr.$$

Furthermore, recall that $|Q_T(-\infty)| \simeq |u(-\infty)|^2$, which can be estimated by (2.7). Then we could absorb the small terms to the left hand side and reach the following estimate

$$\int_{r_+}^{\infty} |R(r)|^2\, dr \lesssim \int_{r_+}^{\infty} |\hat{F}(r)|^2\, dr.$$

Combining with (2.7) again, we summarize the result into the following proposition.

Proposition 2.13. *Consider $(\omega, m, l) \in \mathscr{A}$, u solving (1.8) with a smooth compactly supported inhomogeneity H. The following estimates hold*

$$|u(-\infty)|^2 \lesssim_{\mathscr{A}} \int_{r_+}^{\infty} |\hat{F}(r)|^2\, dr. \tag{2.11}$$

We are then ready to show the quantitative bound for $|W|^{-1}$.

Lemma 2.14. *Let $H(x^*)$ be compactly supported. For any $(\omega, m, l) \in \mathscr{A}$, define*

$$u(r^*) := W^{-1}\left(u_{out}(r^*)\int_{-\infty}^{r^*} u_{hor}(x^*)H(x^*)dx^* + u_{hor}(r^*)\int_{r^*}^{\infty} u_{out}(x^*)H(x^*)dx^*\right).$$

Then

$$u'' + \left(\omega^2 - V\right)u = H,$$

and u satisfies the boundary conditions of a mode solution.

Putting $r^* = -\infty$, we obtain

$$|u(-\infty)|^2 = |W|^{-2} \left| \int_{-\infty}^{\infty} u_{\text{out}}(x^*) H(x^*) dx^* \right|^2.$$

This, together with the preceding proposition, implies that

$$|W|^{-2} \lesssim \frac{\int_{r_+}^{\infty} \left|(r^2+a^2)^{3/2}\Delta^{-1}H\right|^2 \, dr}{\left|\int_{-\infty}^{\infty} u_{out}(x^*) H(x^*)\, dx^*\right|^2}$$

with a Wronskian independent right hand side. Choosing H compactly supported will give a desired finite bound for W^{-1}.

Remark 2.15. In [Shl15], the quantitative version of mode stability is used to prove ILED in the bounded frequency regime.

2.6. **Integral transformation for subextremal Kerr.** This part is devoted to the analysis of the integral transformation (2.4). We prove Proposition 2.4, 2.5 and 2.6.

Recall the definition of Whiting's integral transformation (2.4) and we denote

$$\begin{aligned}\tilde{g}(z) &= \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(z-r_-)(r-r_-)} \, (r-r_-)^{\eta} \, (r-r_+)^{\xi} \, e^{-i\omega r} R(r) \, dr \\ &= \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(z-r_-)(r-r_-)} \, (r-r_-)^{2\eta} \, (r-r_+)^{2\xi} \, e^{-2i\omega r} g(r) \, dr,\end{aligned}$$

where

$$g(r) := (r-r_-)^{-\eta} \, (r-r_+)^{-\xi} \, e^{i\omega r} R(r).$$

Recall that the equation for R has regular singularity at $r = r_+$ and $r = r_-$ with order $(r-r_+)^{\xi}$ and $(r-r_+)^{\eta}$, respectively, and this explains the first two factors in the definition of g. Moreover, we notice that $R \sim e^{i\omega r^*}/r$ and recall that $r^* \sim r + 2M \ln r$ as $r \to \infty$, so we know that

$$R \sim e^{i\omega r} r^{2iM\omega - 1}.$$

Then one can verify that g satisfies the confluent Heun equation

$$\frac{d^2 g}{dr^2} + \left(\frac{\gamma}{r-r_-} + \frac{\delta}{r-r_+} + p \right) \frac{dg}{dr} + \left(\frac{\alpha p}{r-r_+} + \frac{\sigma}{(r-r_+)(r-r_-)} \right) g = G, \tag{2.12}$$

where

$$\gamma = 2\eta + 1 =: \gamma_0, \quad \delta = 2\xi + 1 =: \delta_0, \quad p = -2i\omega =: p_0, \quad \alpha = 1 =: \alpha_0,$$
$$\sigma = 2am\omega - 2\omega r_- i - \lambda_{\omega ml} - a^2\omega^2 =: \sigma_0, \quad G = (r-r_+)^{-\xi}(r-r_-)^{-\eta} e^{i\omega r} \hat{F} =: G_0.$$

Then one can verify the equation for $\tilde{g}$ via integration by parts. It turns out that $\tilde{g}$ satisfies the confluent Heun equation with

$$\gamma = \alpha_0, \quad \delta = \gamma_0 + \delta_0 - \alpha_0, \quad p = p_0, \quad \alpha = \gamma_0, \quad \sigma = \sigma_0,$$

that is,

$$\frac{\partial^2 \tilde{g}}{\partial x^2} + \left(\frac{1}{z-r_-} + \frac{1-4iM\omega}{z-r_+} - 2i\omega \right) \frac{\partial \tilde{g}}{\partial x} + \left(\frac{-2i\omega(2\eta+1)}{z-r_+} + \frac{2am\omega - 2\omega r_- i - \lambda_{\omega ml} - a^2\omega^2}{(z-r_+)(z-r_-)} \right) \tilde{g} = \tilde{G}$$

for $\operatorname{Im} z \geq 0$, where $\tilde{G}$ is the tilde transform (the transform from g to $\tilde{g}$) of G_0. In particular, when $z = x \in (r_+, \infty)$, it is smooth as a function on this interval.

Remark 2.16. The integral transformation with suitable kernel taking a confluent Heun equation to another one with mixed coefficients has been studied historically. However, there is no good reason why this specific kernel in Whiting's choice makes the equation satisfy nice properties so that the proof analogous to Schwarzschild case works out.

Then one can pass the equation for $\tilde{g}$ to the equation for $\tilde{u}$ by a direct but tedious computation. Furthermore, a boundary behavior analysis for $\tilde{g}$ is needed. We claim that the following is the key lemma.

Lemma 2.17. *Let h be a smooth function on $[r_+, \infty)$ which vanishes in $[r_+ + 2, \infty)$. Recall that we previously defined*

$$\xi := \frac{i(am - 2Mr_+\omega)}{r_+ - r_-} \in i\mathbb{R}.$$

For $\nu > 0$, define

$$Z(\nu) := Z(\nu, 0) = \int_{r_+}^{\infty} e^{i\nu r} (r - r_+)^{2\xi} h(r) dr.$$

Then we have

$$Z(\nu) = \exp\left(\frac{i\pi}{2}(1 + 2\xi)\right) \Gamma(2\xi + 1) h(r_+) e^{i\nu r_+} \nu^{-1-2\xi} + O\left(\nu^{-2}\right) \text{ as } \nu \to \infty$$

where

$$\Gamma(z) := \int_0^{\infty} e^{-t} t^{z-1} dt$$

is the Gamma function.

2.7. **Mode stability of Teukolsky equation with extremal Kerr parameters.** For completeness, we mention some further generalizations of [Shl15]. [And+17] considered the mode stability of Teukolsky equation (1.16) on the real axis for all spins. In the extremal case, for $s = 0, \pm 1, \pm 2$, it was settled by [Tei20] using a variant of Whiting's integral transform

$$\begin{aligned} \tilde{u}(x) := \lim_{y \to 0} & (x^2 + 2M^2)^{1/2} (x - M)^{-s} (x - 2M)^{\alpha} \times \\ & \times \int_M^{\infty} e^{-\frac{2\gamma}{M}(x + iy - M)(r - M)} (r - M)^{\alpha} e^{\beta (r - M)^{-1}} e^{-\gamma r} R(r) dr, \end{aligned} \tag{2.13}$$

and reduces the consideration to $s \in \frac{1}{2}\mathbb{Z}_{\leq 0}$ thanks to the Teukolsky–Starobinsky identities.

Later in [CT22], some hidden spectral symmetries for Teukolsky equations were found and one of which is directly related to the Whiting's integral transformation (2.4).

To study the radial ODE (1.19), [CT22, Lemma 2.6] reveals that (1.19) can be written as another ODE with parameters $m_1, m_2, m_3, E, p \in \mathbb{C}$ of the following form :

$$\left[z(z-1)\frac{d^2}{dz^2} - p^2 z(z-1) - m_3\, p\,(2z-1) + (E + \frac{1}{4}) - \frac{m_1 m_2}{z} - \frac{[(m_1 + m_2)^2 - 1]}{4z(z-1)}\right] y(z) = 0 \tag{2.14}$$

by writing $z := \frac{r-r_-}{r_+-r_-}$ and choosing

$$m_1 := s - \xi - \eta = s + 2iM\omega\,, \qquad m_2 := \eta - \xi = i\frac{2M^2\omega - am}{\sqrt{M^2-a^2}} = \frac{i}{2}\left(\frac{\omega - m\omega_+}{\kappa_+} + \frac{\omega - m\omega_-}{\kappa_-}\right)\,,$$
$$m_3 := -s - \xi - \eta = -s + 2iM\omega\,,$$
$$p := 2i\omega\sqrt{M^2-a^2}\,, \qquad E := -\bar{\lambda} - a^2\omega^2 - s^2 + 8M^2\omega^2 - \frac{1}{4}\,. \tag{2.15}$$

It was proved in [CT22, Proposition 2.8] that the existence of nontrivial solutions (with suitable boundary conditions) to (2.14) is equivalent to the existence of such point spectrum to an ODE with m_1, m_2 replaced by m_2, m_3 or m_3, m_1 in (2.14). The $m_2 \leftrightarrow m_3$ symmetry reveals the Whiting's transformation. In fact, after making this symmetry transform, and then multiplying the function by $(r^2+a^2)^{\frac{1}{2}}/\Delta^{\frac{1}{2}}$ will give the Whiting's transformation.

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