[1]

TWO-ENDS FURSTENBERG CONJECTURE

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This is a note on [Coh25].

1. Preliminaries

We first recall some definitions from [WW24, Section 1].

Definition 1.1 (Shading)**.** Let L be a family of lines in $\mathbb{R}^n$ and let $\delta \in (0,1)$. A **shading** $Y : L \to B^n(0,1)$ is an assignment such that $Y(\ell) \subset N_\delta(\ell) \cap B^n(0,1)$ is a union of δ-balls in $\mathbb{R}^n$ for all $\ell \in L$. We writes $(L,Y)_\delta$ to emphasize the dependence on δ.

Similarly, given a family of δ-tubes $\mathcal{T}$, a **shading** $Y : \mathcal{T} \to \mathbb{R}^n$ is an assignment such that $Y(T) \subset T$ is a union of δ-balls in $\mathbb{R}^n$ for all $T \in \mathcal{T}$.

We make a few comments on the definition.

- Note that we will use the notation δ-ball to denote a ball of radius δ, same as a δ-cube.
- Here, a shading is required to be a union of δ-balls so that the smallest width it can be is δ. Note that there are no requirements on how these balls intersect.
- In Kakeya setting, it is usually helpful to think that $Y(T) = T$ since there will be no requirement on the spacing of the shading.
- The motivation of considering such δ-tubes come from wave packet decomposition and Fourier restriction problems. For g_θ supported on θ (a $R^{-\frac{1}{2}}$-cap on a curved surface), we have $E_S g_\theta$ essentially constant on tubes of size $R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}} \times R$ with the same dual direction of θ. For any f in physical side, we have wave packet decomposition $f = \sum_T f_T$ with f_T essentially supported in T. Therefore, one important aspect of Fourier restriction is to understand how tubes interact with each other.

Definition 1.2. Let $\delta \in (0,1)$ and let $(L,Y)_\delta$ be a set of lines and shading. We say Y is **λ-dense**, if $|Y(\ell)| \geq \lambda |N_\delta(\ell)|$.

Definition 1.3 (Two-ends)**.** Let $(L,Y)_\delta$ be a set of lines and shading. Let $0 < \epsilon_2 < \epsilon_1 < 1$. We say Y is $(\epsilon_1, \epsilon_2, C)$**-two-ends** if for all $\ell \in L$ and all $\underbrace{\delta \times \cdots \times \delta}_{(n-1)\,\text{copies}} \times \delta^{\epsilon_1}$-tubes $J \subset N_\delta(\ell)$,

$$|Y(\ell) \cap J| \leq C\delta^{\epsilon_2} |Y(\ell)|. \tag{1.1}$$

When the constant C is not important in the context, we say Y is (ϵ_1, ϵ_2)**-two-ends**, or simply **two-ends**. A similar definition applies to a single shading $Y(\ell)$. Similarly, it also applies to $Y(T)$.

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Remark 1.4. In [WW24], it was mentioned that this is the weakest possible requirement on the spacing. This is because all other spacing conditions, such as uniform spacing, require information on all scales. Here, the only information provided is on scale $\frac{1}{K} = \delta^{\epsilon_1}$.

Definition 1.5. Let $\delta \in (0,1)$ be a small number. Let E be a finite union of δ-balls in $\mathbb{R}^n$. For another $\rho \geq \delta$, let

$$|E|_\rho = \min\left\{\#\mathcal{D}_\rho : \mathcal{D}_\rho \text{ is a covering of } E \text{ by } \rho\text{-balls }\right\}.$$

We recall the two-ends Furstenberg conjecture in [WW24, Conjecture 0.9].

Conjecture 1.6. *Let $\delta \in (0,1)$. Let $(L, Y)_\delta$ be a set of directional δ-separated lines in $\mathbb{R}^n$ with an (ϵ_1, ϵ_2) two-ends, λ-dense shading. Then for any $\epsilon > 0$,*

$$|\bigcup_{\ell \in L} Y(\ell)| \geq c_\epsilon \delta^\epsilon \delta^{O(\epsilon_1)} \lambda^{\frac{n-1}{2}} \sum_{\ell \in L} |Y(\ell)|. \tag{1.2}$$

A slightly modified version (with tubes) of the conjecture is stated in [Coh25] in $\mathbb{R}^3$, which will be derived as a corollary of the one above with its tube version.

Conjecture 1.7. *Let $\mathcal{T}$ be a collection of δ-tube segments, the directions of which are δ-separated. For each $T \in \mathcal{T}$, let $Y(T)$ be an (ϵ_1, ϵ_2)-two-ends shading, with $|Y(T)|$ constant over $T \in \mathcal{T}$. Then for all $\epsilon > 0$,*

$$|\bigcup_{T \in \mathcal{T}} Y(T)| \gtrsim_\epsilon \delta^\epsilon \delta^{O(\epsilon_1)} (\sharp\mathcal{T}) |T| \Big(\frac{|Y(T)|}{|T|}\Big)^2.$$

Remark 1.8. In the second version, since $|Y(T)|$ is assumed to be constant for all $T \in \mathcal{T}$, we have $\sum_{T \in \mathcal{T}} |Y(T)| = (\sharp\mathcal{T})|Y(T)|$ and we take $\lambda = \frac{|Y(T)|}{|T|}$, which gives the second conjecture.

We compare the inequality above as [WZ25, Theorem 1.2], which serves as the essential step towards the proof of the Kakeya set conjecture in $\mathbb{R}^3$.

Theorem 1.9 ([WZ25, Theorem 1.2])**.** *For each $\epsilon > 0$, there exists $K \gg 1$ such that the following holds for all δ sufficiently small.*

Given a set of δ-tubes contained in the unit ball in $\mathbb{R}^3$ which are δ-separated in directions. For each $T \in \mathcal{T}$, let $Y(T) \subset T$ be a measurable set with $|Y(T)| \geq \lambda |T|$. Then

$$|\bigcup_{T \in \mathcal{T}} Y(T)| \geq c_\epsilon \delta^\epsilon \lambda^K \sum_{T \in \mathcal{T}} |Y(T)|.$$

Remark 1.10. Kakeya does not have spacing condition on the shading $Y(T)$. In other words, one can take δ^{ϵ_1} to be a constant for comparison. The only requirement is the density condition on the shading.

Note that the discussion above under the condition that the tubes are δ-separated. Instead, [Coh25] shows that this condition is somehow equivalent to a set being Katz–Tao. In $\mathbb{R}^3$, Katz–Tao condition is that the Katz–Tao convex-Wolff constant C_{KT-CW} is bounded.

Definition 1.11 ([WZ25, Definition 1.3])**.** We define $C_{KT-CW}(\mathcal{T})$ as follows:

$$C_{KT-CW}(\mathcal{T}) = \sup_{W \text{ a convex set}} \frac{\#\{T \in \mathcal{T} : T \subset W\}}{|W|/|T|}, \quad \text{where } |T| = \delta^2, \text{ i.e., } \mathcal{T} \text{ is a set of } \delta\text{-tubes in } \mathbb{R}^3.$$

We say that $\mathcal{T}$ obeys the Katz–Tao Convex Wolff Axioms with error $C_{KT-CW}(\mathcal{T})$.

We have a few remarks.

- This is a generalization of the standard non-concentration condition
$$|E \cap B| \leq (r/\delta)^2,$$
where $E \subset \mathbb{R}^n$ is a δ-separated set and B is a ball of radius r.
- Another non-concentration condition named Frostman Slab Wolff was mentioned in [WZ25, Definition 1.3].
- Note that the Katz–Tao condition in $\mathbb{R}^2$ does not require using convex sets and it is different here because of the 3 dimensional setting.

2. Construction of the counterexample

In the procedure, we identify

$$\begin{aligned} &\text{non-horizontal lines in } \mathbb{R}^3 \leftrightarrow \text{points in } \mathbb{R}^4, \\ &\{(a,b,0)+t(c,d,1) : t \in \mathbb{R}\} \leftrightarrow (a,b,c,d). \end{aligned} \tag{2.1}$$

Through this identification, our tube set $\mathbb{T}$ in $\mathbb{R}^3$ are given by the following set $X \subset \mathbb{R}^4$. We fix $b_0, b_1, b_2 \in \mathbb{Z}$ and set $\alpha = \frac{1}{6}$.

$$X := \left\{ \delta^{\frac{1}{4}} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \delta^{\frac{1}{2}} \begin{pmatrix} \delta^\alpha a_0 \\ \delta^\alpha a_1 \\ \delta^\alpha a_2 \\ 0 \end{pmatrix} + s\delta^{\frac{1}{2}} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ 1 \end{pmatrix} : s \in [0,1],\ n_j, a_j \in \mathbb{Z},\ \delta^{\frac{1}{4}} n_j \in [0,1],\ \delta^\alpha a_j \in [0,1] \right\}.$$

Each n_j has $\delta^{-\frac{1}{4}}$ choices and one view them as the centers. Therefore, X is contained in $\delta^{-1} = (\delta^{-\frac{1}{4}})^4$ many $\delta^{\frac{1}{2}}$-balls. Each $\delta^{\frac{1}{2}}$ ball contains a collection of line segments with fixed angle $(b_0, b_1, b_2, 1)$. There are $\delta^{-\frac{1}{2}} = (\delta^{-\alpha})^3$ many such line segments in each $\delta^{\frac{1}{2}}$-ball.

Now we describe how X is connected to a tube set $\mathbb{T}$ in $\mathbb{R}^3$ via the identification (2.1). We obtain a family of lines with parameter t:

$$\delta^{\frac{1}{4}} \begin{pmatrix} n_1 + tn_3 \\ n_2 + tn_4 \\ 0 \end{pmatrix} + \delta^{\frac{1}{2}} \begin{pmatrix} \delta^\alpha (a_0 + ta_2) \\ \delta^\alpha a_1 \\ 0 \end{pmatrix} + s\delta^{\frac{1}{2}} \begin{pmatrix} b_0 + tb_2 \\ b_1 + t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}.$$

By considering the z-axis range $z \in [0,1]$ (i.e., $t \in [0,1]$), we can thicken such line segments to δ-tubes and denote them by $\mathbb{T}$.

Therefore, one can argue similarly as above to observe that $\mathbb{T}$ is contained in δ^{-1} many $\delta^{\frac{1}{2}}$-tubes (i.e., tubes of size $\delta^{\frac{1}{2}} \times \delta^{\frac{1}{2}} \times 1$). We denote such $\delta^{\frac{1}{2}}$-tubes by $\mathbb{T}^{\delta^{\frac{1}{2}}}$ and for each $T^{\delta^{\frac{1}{2}}} \in \mathbb{T}^{\delta^{\frac{1}{2}}}$, we use $\mathbb{T}[T^{\delta^{\frac{1}{2}}}]$ to denote the δ-tubes contained in $T^{\delta^{\frac{1}{2}}}$.

We consider a single line segment in $\mathbb{R}^4$ described above. For simplicity, we assume that X contains a line segment of the form

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\delta^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : s \in [0,1] \right\}.$$

Via (2.1), taking $\widetilde{s} = s\delta^{\frac{1}{2}}$, we obtain a surface which is swept through by the core line of tubes:

$$\{(\widetilde{s}, 0, 0) + \mathbb{R}(0, \widetilde{s}, 1) : \widetilde{s} \in [0, \delta^{\frac{1}{2}}]\} = \{(x, xz, z) : x \in [0, \delta^{\frac{1}{2}}], z \in [0,1]\},$$

which is a regulus.

Finally, we equip the tubes with appropriate shading to make it (ϵ_1, ϵ_2)-two-ends. We take a shading that is a constant subset of the z-axis:

$$Y(T) = \left\{(x, y, z) \in T : z \in Z_{\text{LowHeight}} + \left[0, \delta^{1/2}\right]\right\}$$
$$Z_{\text{LowHeight}} = \frac{1}{K}\mathbb{Z} \cap [0,1],$$

where $K \in \mathbb{Z}$ is fixed such that $K^{-1} = \delta^{\epsilon_1}$. Here, $\epsilon_2 = \frac{1}{2}$ makes it (ϵ_1, ϵ_2)-two-ends.

3. Analyzing the counterexample

3.1. **Analyzing the main estimate.** To justify that $\mathbb{T}$ with the shading described as above falsifies Conjecture 1.7 in 3 dimensions, it suffices to show that

$$|\bigcup_{T \in \mathbb{T}} Y(T)| \lesssim \delta^{\epsilon} \delta^{C\epsilon_1} \frac{|Y(T)|}{|T|} (\sharp \mathbb{T}) |Y(T)| = \delta^{\epsilon} K^{-C} |Y(T)|^2 (\sharp \mathbb{T}) |T|^{-1}.$$

First, we compute

$$\sharp \mathbb{T} = \underbrace{\delta^{-1} \times \delta^{-\frac{1}{2}}}_{\substack{\sharp\text{line segments in } X, \\ \text{we have } \delta^{-1} \text{ many } \delta^{\frac{1}{2}}\text{-balls}, \\ \text{in which each containing } \delta^{-\frac{1}{2}} \text{ many}}} \times \underbrace{\delta^{-\frac{1}{2}}}_{\substack{\text{note that } s \text{ comes with } \delta^{\frac{1}{2}} \text{ in the definition of } X, \\ \text{so to let } s\delta^{\frac{1}{2}} \text{ having } \delta \text{ separation}, \\ \text{there are } \delta^{-\frac{1}{2}} \text{ many choices of } s \\ \text{such that the tubes in } \mathbb{T} \text{ are essentially disjoint}}} = \delta^{-2}.$$

Moreover, $|T|_\delta = \delta^{-1}$ and $|Y(T)|_\delta = K\delta^{-\frac{1}{2}}$.

It suffices to establish that

$$|\bigcup_{T \in \mathbb{T}} Y(T)|_\delta \lesssim \delta^{\epsilon} K^{-C} |Y(T)|_\delta^2 (\sharp \mathbb{T}) |T|_\delta^{-1} = \delta^{\epsilon} K^{2-C} \delta^{-2}.$$

Now we try to compute $|\bigcup_{T \in \mathbb{T}} Y(T)|_\delta$ by slicing it via $\{z = t\}$. Recall the construction of $Y(T)$, we take $t = t_0 + \delta^{\frac{1}{2}} t_1$ with $t_0 \in Z_{\text{LowHeight}}$, $t_1 \in [0,1]$. We denote by $\pi_t(\mathbb{T})$

the slice of $\mathbb{T}$ with a plane at height t. This corresponds to the projection $\mathbb{R}^4 \to \mathbb{R}^2$ with $(a,b,c,d) \mapsto (a+tc, b+td)$. Indeed,

$$\begin{aligned}\pi_t(\mathbb{T}) &= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} + t\begin{pmatrix} c \\ d \end{pmatrix} : (a,b,c,d) \in X \right\} \subset \mathbb{R}^2 \\ &= \delta^{1/4}\begin{pmatrix} n_1 + tn_3 \\ n_2 + tn_4 \end{pmatrix} + \delta^{1/2}\begin{pmatrix} \delta^\alpha a_0 + t\delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + \delta^{1/2} s \begin{pmatrix} \delta^\alpha b_0 + \delta^\alpha t b_1 \\ \delta^\alpha b_2 + t \end{pmatrix} \\ &= \delta^{1/4}\begin{pmatrix} n_1 + t_0 n_3 \\ n_2 + t_0 n_4 \end{pmatrix} + \delta^{1/2}\begin{pmatrix} \delta^\alpha a_0 + t_0\delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + \delta^{1/2} t_1 \begin{pmatrix} \delta^{1/4} n_3 \\ \delta^{1/4} n_4 \end{pmatrix} + \delta^{1/2} s \begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}.\end{aligned}$$

Thus, this allows us to compute $|\bigcup_{T\in\mathbb{T}} Y(T)|_\delta$ via slicing:

$$|\bigcup_{T\in\mathbb{T}} Y(T)|_\delta \sim |Y(T)|_\delta |\pi_t(\mathbb{T})|_\delta = K\delta^{-\frac{1}{2}} |\pi_t(\mathbb{T})|_\delta. \tag{3.1}$$

Now it suffices to compute the δ-covering number of $\pi_t(\mathbb{T}) \subset \mathbb{R}^2$. Since

$$|\pi_t(\mathbb{T})|_\delta \times (\sharp\delta\times\delta^{\frac{1}{2}} \text{ line segments per } \delta-\text{ball}) = |\pi_t(\mathbb{T})|_{\delta^{\frac{1}{2}}} \times (\sharp\delta\times\delta^{\frac{1}{2}} \text{ line segments per } \delta^{\frac{1}{2}}-\text{ball}),$$

we write

$$|\pi_t(\mathbb{T})|_\delta = |\pi_t(\mathbb{T})|_{\delta^{\frac{1}{2}}} \times (\sharp\delta\times\delta^{\frac{1}{2}} \text{ line segments per } \delta^{\frac{1}{2}}-\text{ball}) \times (\sharp\delta-\text{balls per } \delta\times\delta^{\frac{1}{2}} \text{ line segments}), \tag{3.2}$$

where the line segments are fixed by our choice and the balls are related to the optimal covering number. Hence, due to optimality,

$$(\sharp\delta - \text{balls per } \delta \times \delta^{\frac{1}{2}} \text{ line segments}) = \delta^{-\frac{1}{2}}.$$

We want to find an upper bound for $|\pi_t(\mathbb{T})|_\delta$ and there is a naturally induced $\delta^{\frac{1}{2}}$-ball covering described as follows, which makes $|\pi_t(\mathbb{T})|_{\delta^{\frac{1}{2}}} \le K^2\delta^{-\frac{1}{2}}$. Namely, following the same discussion as in the previous section, originally the four entries will give $(\delta^{-\frac{1}{4}})^4$ many $\delta^{-\frac{1}{2}}$ tubes but now, we only have two entries and we have

$$m_1 := n_1 + t_0 n_3, \quad m_2 := n_2 + t_0 n_4 \in \frac{1}{K}\mathbb{Z}, \quad \delta^{\frac{1}{4}} n_j \in [0,1],$$

which tells us that the choice of centers are $(K\delta^{-\frac{1}{4}})^2 = K^2\delta^{-\frac{1}{2}}$. Indeed,

$$\pi_t(\mathbb{T}) \subset \bigcup_{m_1,m_2\in\frac{1}{K}\mathbb{Z}\cap[0,1]} B\left(\delta^{\frac{1}{4}}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, 10\delta^{\frac{1}{2}}\right).$$

Now we just need to compute how many $\delta \times \delta^{\frac{1}{2}}$-line segments there are in each $\delta^{\frac{1}{2}}$-ball $B\left(\delta^{\frac{1}{4}}\binom{m_1}{m_2}, 10\delta^{\frac{1}{2}}\right)$.

Now these $\delta \times \delta^{\frac{1}{2}}$ line segments in each fixed $\delta^{\frac{1}{2}}$-ball are given by (after shifting the origin of each ball it is located in back to the origin and rescale the length $\delta^{\frac{1}{2}} \to 1$):

$$\begin{pmatrix} \delta^\alpha a_0 + t_0\delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + t_1\begin{pmatrix} \delta^{1/4} n_3 \\ \delta^{1/4} n_4 \end{pmatrix} + s\begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}.$$

We want to compute the number of such line segments, which has the same direction vector

$$\begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}.$$

Consider its normal direction

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} := \begin{pmatrix} -b_1 - t_0 \\ b_0 + t_0 b_2 \end{pmatrix},$$

we claim that the number of such line segments is given by the number of possible outcomes of the following expression when a_0, a_1, a_2, n_3, n_4 are varying:

$$\begin{aligned} &\left\langle \begin{pmatrix} \delta^\alpha a_0 + t_0 \delta^\alpha a_2 \\ \delta^\alpha a_1 \end{pmatrix} + t_1 \begin{pmatrix} \delta^{1/4} n_3 \\ \delta^{1/4} n_4 \end{pmatrix} + s \begin{pmatrix} b_0 + t_0 b_2 \\ b_1 + t_0 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \\ &= \delta^\alpha \left(q_1 a_0 + q_1 t_0 a_2 + q_2 a_1 \right) + t_1 \delta^{1/4} \left(q_1 n_3 + q_2 n_4 \right). \end{aligned}$$

The reason is that we only count things of length $\delta^{\frac{1}{2}}$ in a $\delta^{\frac{1}{2}}$-disk, so as long as the two corresponding lines (rather than line segments) are the same lines, then the line segments are essentially the same one.

Together with $q_1, q_2 \in \frac{1}{CK}\mathbb{Z}$, we have

$$\begin{aligned} \delta^\alpha a_j \in [0,1], \Rightarrow \delta^\alpha (q_1 a_0 + q_1 t_0 a_2 + q_2 a_1) &\in \frac{\delta^\alpha}{CK}\mathbb{Z} \cap [-C, C], \\ \delta^{\frac{1}{4}} n_j \in [0,1] \Rightarrow \delta^{\frac{1}{4}} (q_1 n_3 + q_2 n_4) &\in \frac{1}{CK}\mathbb{Z} \cap [-C, C]. \end{aligned}$$

Thus, we know that we have at most $(CK\delta^{-\alpha}) \times (CK\delta^{-\frac{1}{4}})$ many possibilities.

Finally, we derive from the double counting formula (3.2) that

$$|\pi_t(\mathbb{T})|_\delta \le K^2 \delta^{-\frac{1}{2}} \times C^2 K^2 \delta^{-\frac{1}{4} - \alpha} \times \delta^{-\frac{1}{2}} \le C^2 K^4 \delta^{-1-\frac{5}{12}}.$$

Hence, it follows from (3.1) that

$$|\bigcup_{T \in \mathbb{T}} Y(T)|_\delta \le C^2 K^5 \delta^{-1-\frac{11}{12}} \gtrsim K^{2-C} \delta^{\epsilon - 2}, \text{ for } \delta, \epsilon \text{ small enough},$$

which falsifies Conjecture 1.7 and hence also Conjecture 1.6.

3.2. **Directional δ-separated assumption.** However, there is one missing point that we still need to justify, namely that the assumption in Conjecture 1.7 that tubes in $\mathbb{T}$ are directional δ-separated. It is shown in [Coh25, Section 1] that it suffices to show that $\mathbb{T}$ is Katz–Tao Convex Wolff and hence the directional δ-separated version is achieved via suitable projective transforms. The verification of Katz–Tao Convex Wolff also takes advantage of the two-scale structure that $\mathbb{T}$ is covered by $\mathbb{T}^{\delta^{\frac{1}{2}}}$.

As discussed in last section, out tube set $\mathbb{T}$ is contained in a union of δ^{-1}-many $\delta^{\frac{1}{2}}$-tubes, denoted by $\mathbb{T}^{\delta^{\frac{1}{2}}}$. Recall that $\sharp\mathbb{T}$ is δ^{-2} so each $\mathbb{T}^{\delta^{\frac{1}{2}}}$ contains δ^{-1}-many δ-tubes. (This can be also seen from a more direct computation and this is closely related to the choice $\alpha = \frac{1}{6}$.)

We recall the following statement:

Lemma 3.1 ([WZ25, Lemma 4.12]). *Let $0 < \delta \le \rho \le 1$. Let $\mathcal{T}$ be a set of δ-tubes and $\mathcal{T}_\rho$ be a covering of $\mathcal{T}$, then*

$$C_{KT-CW}(\mathcal{T}) \lesssim \left(\sup_{T_\rho \in \mathcal{T}_\rho} C_{KT-CW}\left(\mathcal{T}^{T_\rho}\right) \right) \left(C_{KT-CW}\left(\mathcal{T}_\rho\right) \right).$$

Take $\rho = \delta^{\frac{1}{2}}$ and $\mathcal{T} = \mathbb{T}$, $\mathcal{T}_\rho = \mathbb{T}^{\delta^{\frac{1}{2}}}$, we reduce the problem to proving the two sets $\mathbb{T}^{\delta^{\frac{1}{2}}}$ and $\mathbb{T}[T^{\delta^{\frac{1}{2}}}]$ are Katz–Tao Convex Wolff.

Since our tube sets are based on a construction in $\mathbb{R}^4$, we would like an interpretation of the Katz–Tao Convex Wolff in terms of $\mathbb{R}^4$. We refer the details to [Coh25, Section 4].

References

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