A Matrix Expander Chernoff Bound

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Vanilla Chernoff Bound

**Thm** [Hoeffding, Chernoff]. If $X_1, \ldots, X_k$ are independent mean zero random variables with $|X_i| \leq 1$ then

$$
P \left[ \left| \frac{1}{k} \sum_i X_i \right| \geq \epsilon \right] \leq 2 \exp(-k\epsilon^2/4)$$
**Vanilla Chernoff Bound**

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\right|
\geq \epsilon
\right]
\leq 2 \exp\left(-k \epsilon^2 / 4\right)
\]

**Two Extensions:**

1. **Dependent** Random Variables
2. Sums of random **matrices**
Thm[Gillman’94]: Suppose $G = (V, E)$ is a regular graph with transition matrix $P$ which has second eigenvalue $\lambda$. Let $f: V \to \mathbb{R}$ be a function with $|f(v)| \leq 1$ and $\sum_v f(v) = 0$. 
Expander Chernoff Bound [AKS’87, G’94]

**Thm[Gillman’94]:** Suppose $G = (V, E)$ is a regular graph with transition matrix $P$ which has second eigenvalue $\lambda$. Let $f : V \rightarrow \mathbb{R}$ be a function with $|f(v)| \leq 1$ and $\sum_v f(v) = 0$. Then, if $v_1, \ldots, v_k$ is a stationary random random walk:

$$\mathbb{P} \left[ \left| \frac{1}{k} \sum_{i} f(v_i) \right| \geq \epsilon \right] \leq 2 \exp(-c(1 - \lambda)k\epsilon^2)$$

Impiles walk of length $k \approx (1 - \lambda)^{-1}$ concentrates around mean.
Intuition for Dependence on Spectral Gap

1 - \lambda = \frac{1}{n^2}
Intuition for Dependence on Spectral Gap

$$1 - \lambda = \frac{1}{n^2}$$

Typical random walk takes $\Omega(n^2)$ steps to see both $\pm 1$. 
Thm[Gillman’94]: Suppose $G = (V, E)$ is a regular graph with transition matrix $P$ which has second eigenvalue $\lambda$. Let $f: V \to \mathbb{R}$ be a function with $|f(v)| \leq 1$ and $\sum_v f(v) = 0$. Then, if $v_1, \ldots, v_k$ is a stationary random walk:

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To generate $k$ indep samples from $[n]$ need $k \log n$ bits. If $G$ is 3-regular with constant $\lambda$, walk needs: $\log n + k \log(3)$ bits.
**Derandomization Motivation**

**Thm[Gillman’94]:** Suppose $G = (V, E)$ is a regular graph with transition matrix $P$ which has second eigenvalue $\lambda$. Let $f : V \to \mathbb{R}$ be a function with $|f(v)| \leq 1$ and $\sum_v f(v) = 0$. Then, if $v_1, \ldots, v_k$ is a stationary random walk:

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To generate $k$ indep samples from $[n]$ need $k \log n$ bits. If $G$ is 3-regular with constant $\lambda$, walk needs: $\log n + k \log(3)$ bits.

When $k = O(\log n)$ reduces randomness \textit{quadratically}.

\textit{Can completely derandomize in polynomial time.}
Matrix Chernoff Bound

**Thm** [Rudelson’97, Ahlswede-Winter’02, Oliveira’08, Tropp’11...]. If $X_1, \ldots, X_k$ are independent mean zero random $d \times d$ Hermitian matrices with $\|X_i\| \leq 1$ then

$$
P \left[ \left\| \frac{1}{k} \sum_i X_i \right\| \geq \epsilon \right] \leq 2d \exp(-k\epsilon^2 / 4)$$
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P \left[ \left\| \frac{1}{k} \sum_i X_i \right\| \geq \epsilon \right] \leq 2d \exp \left( -k \epsilon^2 / 4 \right)
\]

Factor \( d \) is tight because of the diagonal case.

Very generic bound (no independence assumptions on the entries). Many applications + martingale extensions (see Tropp).
**Conj[Wigderson-Xiao’05]:** Suppose $G = (V, E)$ is a regular graph with transition matrix $P$ which has second eigenvalue $\lambda$. Let $f : V \rightarrow \mathbb{C}^{d \times d}$ be a function with $||f(v)|| \leq 1$ and $\sum_v f(v) = 0$. Then, if $v_1, \ldots, v_k$ is a stationary random walk:

$$\mathbb{P} \left[ \left\| \frac{1}{k} \sum_i f(v_i) \right\| \geq \epsilon \right] \leq 2d \exp(-c(1 - \lambda)k\epsilon^2)$$

Motivated by derandomized Alon-Roichman theorem.
Main Theorem

**Thm.** Suppose $G = (V, E)$ is a regular graph with transition matrix $P$ which has second eigenvalue $\lambda$. Let $f: V \to \mathbb{C}^{d \times d}$ be a function with $\|f(v)\| \leq 1$ and $\sum_v f(v) = 0$. Then, if $v_1, \ldots, v_k$ is a stationary random walk:

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Gives black-box derandomization of any application of matrix Chernoff
1. Proof of Chernoff: reduction to mgf

**Thm [Hoeffding, Chernoff].** If $X_1, \ldots, X_k$ are independent mean zero random variables with $|X_i| \leq 1$ then

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$$\mathbb{P} \left[ \sum_i X_i \geq k\epsilon \right] \leq e^{-tk\epsilon} \mathbb{E} \exp \left( t \sum_i X_i \right)$$

Markov
1. Proof of Chernoff: reduction to mgf

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Indep.
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$$\leq e^{-tk\epsilon} (1 + t\mathbb{E}X_i + t^2)^k$$

Bounded
1. Proof of Chernoff: reduction to mgf

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$$\leq e^{-tk\epsilon} (1 + t^2)^k$$
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$$

$$
\leq e^{-tk\epsilon} (1 + t^2)^k \leq e^{-tk\epsilon + kt^2}
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1. Proof of Chernoff: reduction to mgf

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\mathbb{P} \left[ \left| \frac{1}{k} \sum_{i} X_i \right| \geq \varepsilon \right] \leq 2 \exp\left( -\frac{k\varepsilon^2}{4} \right)
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\mathbb{P} \left[ \sum_{i} X_i \geq k\varepsilon \right] \leq e^{-tk\varepsilon} \mathbb{E} \exp\left( t \sum_{i} X_i \right) = e^{-tk\varepsilon} \prod_{i} \mathbb{E} e^{tx_i}
$$

$$
\leq e^{-tk\varepsilon} (1 + ... + t^2)^k \leq e^{-tk\varepsilon + kt^2} \leq e^{-\frac{k\varepsilon^2}{4}}
$$
2. Proof of Expander Chernoff

**Goal:** Show $\mathbb{E} \exp (t \sum_i f(v_i)) \leq \exp (c \lambda k t^2)$
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**Goal:** Show $\mathbb{E} \exp(t \sum_i f(v_i)) \leq \exp(c_\lambda k t^2)$

**Issue:** $\mathbb{E} \exp(\sum_i f(v_i)) \neq \prod_i \mathbb{E} \exp(tf(v_i))$

How to control the mgf without independence?
Step 1: Write mgf as quadratic form

\[ \mathbb{E} e^t \sum_{i \leq k} f(v_i) \]

\[ = \sum_{i_0, \ldots, i_k \in V} P(v_0 = i_0) P(i_0, i_1) \cdots P(i_{k-1}, i_k) \exp \left( t \sum_{1 \leq j \leq k} f(i_j) \right) \]
Step 1: Write mgf as quadratic form

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$$= \sum_{i_0, \ldots, i_k \in V} \mathbb{P}(v_0 = i_0) P(i_0, i_1) \cdots P(i_{k-1}, i_k) \exp\left(t \sum_{1 \leq j \leq k} f(i_j)\right)$$

$$= \frac{1}{n} \sum_{i_0, \ldots, i_k \in V} P(i_0, i_1) e^{tf(i_1)} \cdots P(i_{k-1}, i_k) e^{tf(i_k)}$$
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\[ = \langle u, (EP)^k u \rangle \text{ where } u = \left( \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right). \]

\[ E = \begin{bmatrix} e^{tf(1)} \\ \vdots \\ e^{tf(n)} \end{bmatrix} \]
Step 2: Bound quadratic form

**Goal:** Show $\langle u, (EP)^k u \rangle \leq \exp(c\lambda kt^2)$

**Observe:** $||P - J|| \leq \lambda$ where $J =$ complete graph with self loops
So for small $\lambda$, should have $\langle u, (EP)^k u \rangle \approx \langle u, (EJ)^k u \rangle$
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**Approach 1.** Use perturbation theory to show $||EP|| \leq \exp(c_\lambda t^2)$
Step 2: Bound quadratic form

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So for small $\lambda$, should have $\langle u, (EP)^k u \rangle \approx \langle u, (EJ)^k u \rangle$

**Approach 1.** Use perturbation theory to show $\|EP\| \leq \exp(\lambda t^2)$

**Approach 2.** (Healy’08) track projection of iterates along $u$. 
Simplest case: $\lambda = 0$
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$EP\nu$
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$EPEP\nu$
**Observations**

**Observe:** $P$ shrinks every vector orthogonal to $u$ by $\lambda$.

$$\langle u, E u \rangle = \frac{1}{n} \sum_{v \in V} e^{tf(v)} = 1 + t^2$$ by mean zero condition.
A small dynamical system

\[ m_\perp = \|Q_\perp v\| \]

\[ m_\parallel = \langle v, u \rangle \]

\[ v = (EP)^j u \]
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\[ e^{t^2} \]

\[ m_\parallel \]

\[ m_\perp \]

\[ \lambda = 0 \]

\[ \nu \sim EP\nu \]
A small dynamical system

\[ m_{\perp} = \|Q_{\perp}v\| \]

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\[ m_{\parallel} = \langle v, u \rangle \]

\[ m_{\parallel} = e^{t^2} \]

\[ m_{\parallel} = e^{t\lambda} \]

\[ \lambda t \]

any \( \lambda \)

\[ v \sim EPv \]
A small dynamical system

\[ m_{\perp} = \|Q_{\perp}v\| \]

\[ v = (EP)^j u \]

\[ m_{\parallel} = \langle v, u \rangle \]

\[ t \leq \log \frac{1}{\lambda} \]

\[ \lambda t \leq 1 \]

\[ v \sim EPv \]
A small dynamical system

\[ m_\perp = \|Q_\perp v\| \]

\[ v = (EP)^j u \]

\[ m_\parallel = \langle v, u \rangle \]

Any mass that leaves gets shrunk by \( \lambda \)
Analyzing dynamical system gives

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Generalization to Matrices?

Setup: $f: V \rightarrow \mathbb{C}^{d \times d}$, random walk $v_1, \ldots, v_k$.

Goal:

$$\mathbb{E}\text{Tr} \left[ \exp \left( t \sum_i f(v_i) \right) \right] \leq d \cdot \exp(ckt^2)$$

where $e^A$ is defined as a power series.
**Generalization to Matrices?**

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**Main Issue:** \( \exp(A + B) \neq \exp(A) \exp(B) \) unless \([A, B] = 0\)

can’t express \( \exp(\text{sum}) \) as iterated product.
The Golden-Thompson Inequality

Partial Workaround [Golden-Thompson’65]:

\[ Tr(\exp(A + B)) \leq Tr(\exp(A) \exp(B)) \]

Sufficient for independent case by induction.
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Sufficient for **independent** case by induction. For **expander** case, need this for \( k \) matrices.
The Golden-Thompson Inequality

Partial Workaround [Golden-Thompson’65]:

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Sufficient for **independent** case by induction.
For **expander** case, need this for \( k \) matrices. False!

\[ \text{Tr}(e^{A+B+C}) > 0 > \text{Tr}(e^A e^B e^C) \]
Key Ingredient

[Sutter-Berta-Tomamichel’16] If $A_1, \ldots, A_k$ are Hermitian, then

$$\log Tr(e^{A_1+\ldots+A_k}) \leq \int d\beta(b) \log Tr \left[ \left( e^{\frac{A_1(1+ib)}{2}} \ldots e^{\frac{A_k(1+ib)}{2}} \right) \left( e^{\frac{A_1(1+ib)}{2}} \ldots e^{\frac{A_k(1+ib)}{2}} \right)^* \right]$$

where $\beta(b)$ is an explicit probability density on $\mathbb{R}$.
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$$\log \text{Tr}(e^{A_1^+ \cdots + A_k})$$

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1. Matrix on RHS is always PSD.
2. **Average-case** inequality: $e^{A_i/2}$ are conjugated by unitaries.
3. Implies Lieb’s concavity, triple-matrix, ALT, and more.
Proof of SBT: Lie-Trotter Formula

\[ e^{A+B+C} = \lim_{\theta \to 0^+} \left( e^{\theta A} e^{\theta B} e^{\theta C} \right)^{1/\theta} \]
Proof of SBT: Lie-Trotter Formula

\[ e^{A+B+C} = \lim_{\theta \to 0^+} (e^{\theta A} e^{\theta B} e^{\theta C})^{1/\theta} \]

\[ \log \text{Tr} \ e^{A+B+C} = \lim_{\theta \to 0^+} 2\log ||G(\theta)||_{2/\theta}/\theta \]

For \( G(z) := e^{zA/2} e^{zB/2} e^{zC/2} \)
Complex Interpolation (Stein-Hirschman)

\[ \log ||G(\theta)||_{2/\theta} \]
Complex Interpolation (Stein-Hirschman)

For each $\theta$, find analytic $F(z)$ st:

$|F(it)| = 1$

$|F(1 + it)| \leq \|G(1 + it)\|_2$

$|F(\theta)| = \|G(\theta)\|_{2/\theta}$
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$$|F(it)| = 1$$

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$$|F(\theta)| = \|G(\theta)\|_{2/\theta}$$

$$\log|F(\theta)| \leq \int \log |F(it)| + \int \log |F(1 + it)|$$
Complex Interpolation (Stein-Hirschman)

For each $\theta$, find analytic $F(z)$ st:

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$$\lim_{\theta \to 0}$$

$$\log |F(\theta)| \leq \int \log |F(it)| + \int \log |F(1 + it)|$$
[Sutter-Berta-Tomamichel’16] If $A_1, \ldots, A_k$ are Hermitian, then

$$\log \text{Tr} (e^{A_1^+ \ldots + A_k})$$

$$\leq \int d\beta(b) \log \text{Tr} \left[ \left( e^{\frac{A_1(1+ib)}{2}} \ldots e^{\frac{A_k(1+ib)}{2}} \right) \left( e^{\frac{A_1(1+ib)}{2}} \ldots e^{\frac{A_k(1+ib)}{2}} \right)^* \right]$$

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[Sutter-Berta-Tomamichel’16] If $A_1, \ldots, A_k$ are Hermitian, then

$$\log \text{Tr}(e^{A_1} \cdots e^{A_k}) \leq \int d\beta(b) \log \text{Tr} \left[ \left( e^{\frac{A_1(1+ib)}{2}} \cdots e^{\frac{A_k(1+ib)}{2}} \right) \left( e^{\frac{A_1(1+ib)}{2}} \cdots e^{\frac{A_k(1+ib)}{2}} \right)^* \right]$$

where $\beta(b)$ is an explicit probability density on $\mathbb{R}$.

**Issue.** SBT involves integration over unbounded region, bad for Taylor expansion.
Bounded Modification of SBT

Solution. Prove bounded version of SBT by replacing strip with half-disk.

**[Thm]** If $A_1, \ldots, A_k$ are Hermitian, then

$$ \log \text{Tr}(e^{A_1} + \cdots + e^{A_k}) \leq \int d\beta(b) \log \text{Tr} \left[ \left( e^{\frac{A_1 e^{ib}}{2}} \cdots e^{\frac{A_k e^{ib}}{2}} \right) \left( e^{\frac{A_1 e^{ib}}{2}} \cdots e^{\frac{A_k e^{ib}}{2}} \right)^* \right] $$

where $\beta(b)$ is an explicit probability density on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

**Proof.** Analytic $F(z)$ + Poisson Kernel + Riemann map.
Handling Two-sided Products

Issue. Two-sided rather than one-sided products:

\[
\text{Tr} \left[ \left( e^{\frac{tf(v_1)e^{ib}}{2}} \ldots e^{\frac{tf(v_k)e^{ib}}{2}} \right) \left( e^{\frac{tf(v_1)e^{ib}}{2}} \ldots e^{\frac{tf(v_k)e^{ib}}{2}} \right)^* \right]
\]
Handling Two-sided Products

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\]

**Solution.**
Encode as one-sided product by using \( Tr(AXB) = (A \otimes B^T)vec(X) \):

\[
\langle e^{\frac{tf(v_1)e^{ib}}{2}} \otimes e^{\frac{tf(v_1)^*e^{ib}}{2}} \cdots e^{\frac{tf(v_k)^*e^{-ib}}{2}} \otimes e^{\frac{tf(v_k)^*e^{-ib}}{2}} vec(I_d), vec(I_d) \rangle
\]
Finishing the Proof

Carry out a version of Healy’s argument with $P \otimes I_{d^2}$ and:

$$E = \begin{bmatrix}
\frac{tf(1)e^{ib}}{2} & \frac{tf(1)*T e^{ib}}{2} \\
\frac{e}{2} & \otimes & \frac{e}{2}
\end{bmatrix} \ldots
\begin{bmatrix}
\frac{tf(n)e^{ib}}{2} & \frac{tf(n)*T e^{-ib}}{2} \\
\frac{e}{2} & \otimes & \frac{e}{2}
\end{bmatrix}$$

And $vec(I_d) \otimes u$ instead of $u$.

This leads to the additional $d$ factor.
Main Theorem

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$$\mathbb{P} \left[ \left\| \frac{1}{k} \sum_i f(v_i) \right\| \geq \epsilon \right] \leq 2d \exp(-c(1 - \lambda)k\epsilon^2)$$
Open Questions

Other matrix concentration inequalities
  (multiplicative, low-rank, moments)
Other Banach spaces
  (Schatten norms)
More applications of complex interpolation