Girth, Expansion, and Localization of Graph Eigenfunctions

Nikhil Srivastava
w/ Noga Alon (Princeton) and Shirshendu Ganguly (Berkeley)

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Setup

Undirected $d + 1$-regular graph $G$ on $n$ vertices.

Adjacency matrix $A$ has eigenvalues

$$d + 1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Q. What is the combinatorial meaning of the interior eigenvectors?
A Warmup

**Observation.** If $Av = \lambda v$ and $v$ is supported on $k$ vertices, then

$$girth(A) \leq 4\log_d(k)$$
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**Proof.**

![Graph $G$ with vertices $S$, $+$, $-$, and $0$]
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$$\text{girth}(A) \leq 4\log_d(k)$$

**Proof.**

$$0 = v(x) = \sum_{y \sim x} v(y)$$

Diagram of a graph $G$ with vertices labeled and edges connecting some of them.
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**Proof.**

Every vertex adjacent to $S$ has at least two nbrs in $S$. 
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\[ G \]

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**Proof.**

Every vertex adjacent to $S$ has at least two nbrs in $S$. Replace excursions of length 2 by new edges.
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$$girth(A) \leq 4 \log_d(k)$$

**Proof.**

$$\mindeg(H) \geq d + 1 \Rightarrow girth(H) \leq 2 \log_d(k)$$
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Proof.

$$girth(G) \leq 4\log_d(k)$$

$$k \geq d^{g/4}$$
Localization and Delocalization

**Defn.** A unit vector $v$ is $(\epsilon, k)$-**delocalized** if for all subsets $S \subset [n]$:

$$\|v_S\|_2^2 \geq \epsilon \Rightarrow |S| > k$$

Otherwise it is $(\epsilon, k)$-**localized**, i.e., $\|v_S\|_2^2 \geq \epsilon$ for some $|S| \leq k$. 

**Q.** What about $\epsilon < 1$?
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e.g. $k$-sparse means $(1, k)$-localized

$(\epsilon, 1)$-delocalized implies $||v||_\infty^2 \leq \epsilon$

$||v||_\infty^2 \leq 1/n$ implies $(\epsilon, \epsilon n)$-delocalized for all $\epsilon$. 

**Q.** What about $\epsilon < 1$?
Localization and Delocalization

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$||v||_\infty^2 \leq 1/n$ implies $(\epsilon, \epsilon n)$-delocalized for all $\epsilon$.

**Showed:** If $A$ has girth $g$ then every eigvec is $(d^{g/4}, 1)$-delocalized

**Q.** What about $\epsilon < 1$?
Thm. Suppose $G$ is $d + 1$-regular with girth $g$ and $Av = \lambda v$.

1. If $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ then $v$ is $(\varepsilon, \varepsilon^2 d^{c\varepsilon^2 g})$-delocalized for $\varepsilon \in (0,1)$.

2. If $\lambda \notin (-2\sqrt{d} - \delta, 2\sqrt{d} - \delta)$ then $||v||_\infty^2 \leq d^{-\Omega \delta(g)}$. 
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When $g = \Omega(\log d \, n)$, (1) implies $(\epsilon, \epsilon^2 n^{\Omega(\epsilon^2)})$-deloc and $||v||_\infty^2 \leq (\log d \, n)^{-1/2}$ whereas (2) implies $||v||_\infty^2 \leq n^{-c}$. 

\[
\text{Strong deloc} \quad \text{weak deloc} \quad \text{strong (}\ell_\infty^2 \leq n^{-c}\text{)}
\]

\[
-2\sqrt{d} \text{ (most $\lambda$) } < \sqrt{d}
\]
Questions

**Q1.** How does localization depend on the eigenvalue $\lambda$?

**Q2.** How does localization depend on the mass $\epsilon$?

(in [BL’06], exponent of $\epsilon$ depends on $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ and Diophantine properties of $\lambda$)
Questions

**Q1.** How does localization depend on the eigenvalue $\lambda$?

**Q2.** How does localization depend on the mass $\epsilon$?

(in [BL’06], exponent of $\epsilon$ depends on $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ and Diophantine properties of $\lambda$)

**Q3.** How do high girth graphs compare to random regular graphs?

(cf. [BHY’16] shows bulk eigenvectors have $||v||_\infty^2 \leq \frac{\log^c(n)}{n}$)
Theorem A [Ganguly-S’18]

**Thm.** Suppose $G$ is $d + 1$-regular with girth $g$ and $Av = \lambda v$.

then $v$ is $(\epsilon, \epsilon d^{\frac{\epsilon g}{4} - 3})$-delocalized for $\epsilon \in (0,1)$.

Improved constant and exponent of $\epsilon$ compared to [BL’06] part (1).

**Implies** 

$$||v||_\infty^2 \leq (\log_d n)^{-1}.$$ 

**Contrapositive:** $(k, \epsilon)$-localized implies $g \leq 4 \log(\frac{k}{\epsilon})/\epsilon + O(1)$.

Proof is a technical improvement of [BL’06] (approx. theory + nonbacktracking walks)
Theorem B [Alon-Ganguly-S’19]

Fix \( d \) prime. There is an infinite sequence of \( d + 1 \)-regular graphs \( G_m \) on \( m \) vertices such that:

1. \( \text{girth}(G_m) \geq \left( \frac{1}{3} \right) \log_d(m) \)
2. There is an eigenvector \( A_m \nu = \lambda \nu \) which is \((k, \epsilon)\)–localized for

\[
k = O(d^{4\epsilon \text{girth}(G_m)}) \quad \forall \epsilon \in (0,1]
\]

Implies exponent of \( \epsilon \) in Theorem A cannot be improved.
Theorem B [Alon-Ganguly-S’19]

Fix $d$ prime. There is an infinite sequence of $d + 1$-regular graphs $G_m$ on $m$ vertices such that:

1. \( \text{girth}(G_m) \geq \left( \frac{1}{3} \right) \log_d(m) \)
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The number of such $\lambda$ for each $G_m$ is $\Omega(\log_d(m))$.

The set of $\lambda$ attained by the above sequence is dense in $(-2\sqrt{d}, 2\sqrt{d})$.

Implies arithmetic properties of $\lambda$ do not play a role.
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2. There is an eigenvector $A_m v = \lambda v$ which is $(k, \epsilon)$—localized for

$$k = O(d^{4\epsilon \text{girth}(G_m)}) \quad \forall \epsilon \in (0,1]$$

3. $|\lambda_i(A_m)| \leq 2.12 \sqrt{d}$ for all nontrivial adjacency eigenvalues.

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Proof of Theorem B (simplified)

**Step 1.** Finite $d + 1$-ary tree of depth $\ell$
with $n$ leaves.

**Fact:** has many $(\epsilon, d\epsilon^\ell)$-localized eigenvectors.
equal to zero on the leaves.
Proof of Theorem B

Step 2. Two $d + 1$-ary trees of depth $\ell$, with leaves identified to maximize girth [Erdos-Sachs] or [McKay]

Yields girth $\geq \Omega(\ell) = \Omega(\log_d n)$

Eigenvector equation is satisfied by Reflecting $\psi$ on the paired tree.
Proof of Theorem B

**Step 3.** Let $H$ be a $d + 1$-regular Ramanujan [LPS,Margulis] graph with $n$ defects of degree $d$ at mutual distance $\Omega(\log_d n)$.
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Identify leaves from step 2 with defects.
Retains girth $\Omega(\log_d n)$. 

![Diagram of a graph with nodes labeled $\mathcal{A}$ and $\psi$, and two triangles $T_1$ and $T_2$.](image)
Proof of Theorem B

**Step 3.** Let $H$ be a $d + 1$-regular Ramanujan [LPS,Margulis] graph with $n$ defects of degree $d$ at mutual distance $\Omega(\log_d n)$.

Identify leaves from step 2 with defects. Retains girth $\Omega(\log_d n)$.

Set eigenvector to zero on $H$. 
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Fix $d$ prime. There is an infinite sequence of $d + 1$-regular graphs $G_m$ on $m$ vertices such that:

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Theorem B [Alon-Ganguly-S’19]

Fix $d \geq 3$ prime. There is an infinite sequence of $d + 1$-regular graphs $G_m$ on $m$ vertices such that:

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Spectral Gap

**Observation:** If any eigenvector of $G$ is equal to zero on the interface then

$$v^T A_G v = v^T A_T v + v^T A_H v$$

$$\leq 2\sqrt{d} ||v_T||^2 + 2\sqrt{d} ||v_H||^2 + o(1)$$
Spectral Gap

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$$\leq 2\sqrt{d}||v_T||^2 + 2\sqrt{d}||v_H||^2 + o(1)$$

Key Lemma: Any putative non-Ramanujan eigenvector of $G$ must have at most 5% of its mass on the interface.

(high girth + [Kahale’95] argument)
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\[ \text{\textcolor{red}{\textbf{Note:}}} \] $\leq 2.04$ for $d=2$. But not Ramanujan due to bad vertex expansion [Kohale]
The Quantum Ergodicity Angle

[Shnirelman’74] If the geodesic flow on a compact manifold is ergodic, then there is a dense subsequence of Laplacian eigenfunctions which is equidistributed (a strong notion of delocalization).
The Quantum Ergodicity Angle

[Shnirelman’74] If the geodesic flow on a compact manifold is \textit{ergodic}, then there is a dense subsequence of Laplacian eigenfunctions which is \textit{equidistributed} (a strong notion of delocalization).

QUE Conjecture [Rudnick-Sarnak]: If the manifold is \textit{negatively curved}, this is true for \textbf{all} eigenfunctions. Special case proved by Lindenstrauss.

[Smilansky’07] Study graphs as a simplified model for manifolds.
Quantum Ergodicity on Graphs

[Anantharaman-Le Masson’13] If $G = (V, E)$ is a bounded degree regular high girth expander with unit eigenvectors $v_1, \ldots, v_n$ then:

$$\max_{|f| \leq 1} \sum_i \left| \sum_{x \in V} v_i^2(x) f(x) - \sum_x f(x) \right|^2 = o(n)$$

In fact, they proved this for eigenvectors in any $1/\log(n)$ interval.
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In fact, they proved this for eigenvectors in any $1/\log(n)$ interval.

**Strongest version of QUE:** this is true for every eigenvector.

Theorem B disproves this strongest version.
Questions

• Actual Ramanujan graphs in Theorem B?
  Could Ramanujan + High Girth -> Strong delocalization?

• Minimal assumptions for Quantum Ergodicity on Graphs?
  Construct graphs with many localized $\lambda$ in a small interval?

• More surgery on graphs preserving spectrum [Alon’20], [Paredes’20]

• Use interior eigenvectors to study girth, expansion?
  [cf. Naor’12 for Abelian Cayley Graphs]