Midterm 2 Solutions

1. \[ f(1) = 1, \quad f(k+1) = \sqrt{1+f(k)} \]
   \[ f(2) = \sqrt{1+1} = \sqrt{2}, \text{ known to be irrational.} \]

Let \( P(n) \) be the predicate "\( f(n) \) is irrational".

**Basis Step:** \( P(2) \) says: \( \sqrt{2} \) is irrational, which is true.

**Inductive Step:** Assume \( k \geq 2 \). Assume \( P(k) \) is true. Our goal is to show \( P(k+1) \).
While: \( f(k+1) = \sqrt{1 + f(k)} \).

Assume for the sake of contradiction that \( f(k+1) \) is rational.

Thus, there exist integers \( p, q \neq 0 \) such that
\[
f(k+1) = \frac{p}{q}.
\]
Therefore, \( \sqrt{1 + f(k)} = \frac{p}{q} \).

Squaring both sides, we get
\[
1 + f(k) = \frac{p^2}{q^2}.
\]

Which means
\[
f(k) = \frac{p^2}{q^2} - 1 = \frac{p^2 - q^2}{q^2}.
\]

Which is absurd since by the induction hypothesis, \( f(k) \) is irrational.
2. 0. Formulate conjecture.

Observe: $2, 4, 6, \ldots \text{ any even amount.}$

$5, 10, 15, \ldots$

Can't make 3, can't make 4.

$7 - 5 = 2 \quad \checkmark$

$9 - 5 = 4 \quad \checkmark$

$11 - 5 = 6 \quad \checkmark$

(odd # bigger than 5)

$- 5 = \text{ positive even number!}$

Conjecture: Every positive integer amount except $1$ and $3$ can be made with $2$ and $5$ bills.
Proof: By strong induction.

\[ P(n) = n \text{ $n$ can be made from $2$ and $5$ bills} \]

Goal: \( P(2), P(4), P(5), P(n) \text{ for } n \geq 6 \).

Basis Step: Observe that \( P(2), P(4) \) are true because \( 2, 4 \text{ are even} \), and \( P(5) \) is trivially true.

Inductive Step: Assume \( k \geq 5 \).

Assume \( P(2), P(4), P(5), \ldots, P(k) \) hold.

Goal: \( P(k+1) \) is true.

Observe: \( k+1 = 2 + (k-1) \text{ and } k-1 \geq 4 \).

By the inductive hypothesis, \( P(k-1) \) holds.

So there is a way to write \( k+1 \) in \$2,5\$ bills.
Adding a $2$ ball, we can make $Sk+1$.
This completes the induction.
(3) How many poker hands contain one ace and no king?

By sum rule,

\[ \text{\# hands} = \text{\# hands with one ace, no king} - \text{\# hands with one ace, no king, no hearts}. \]

\[ \text{(A)} \]

\[ \text{(B)} \]

A: Consider the process:

1. Choose one ace, \(4\) ways.
2. Choose 4 more cards, no kings or aces, \(\binom{44}{4}\) ways.

By product rule, \(A = 4 \cdot \binom{44}{4}\).
Process:  
1. Choose one ace
2. Choose 4 cards, no king, ace, heart

\[ 52 - 13 \text{ hearts} = 39 \]
\[ 39 - 3 \text{ kings} - 3 \text{ ace} = 33 \text{ cards} \]

Product rule: \[ 3 \cdot \binom{33}{4} \]

Answer: \[ 4 \cdot \binom{44}{4} - 3 \cdot \binom{33}{4} \]
Given $n$ odd positive, $f: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$.

**Goal:** \((1-f(i)) (2-f(i)) \cdots (n-f(i))\) is even.

**Observe:** Sufficient to show there exists some \(i\) s.t. \((i-f(i))\) is even.

**Observe:** If

- \(i\) and \(f(i)\) are both odd, then \((i-f(i))\) is even!
Let $O = \{k : 1 \leq k \leq n \text{ and } k \text{ is odd}\}$.

**Claim:** There exists some $i \in O$ such that $f(i) \in O$.

**Proof:** By contradiction. Assume not, i.e.

$f(O) \subseteq \{k : k \text{ is even}\} = E$

Observe that $|O| = \lfloor \frac{n+1}{2} \rfloor$, since $n$ is odd. Thus, $f$ maps $\lfloor \frac{n+1}{2} \rfloor$ elements to $|E|$ elements. By pigeonhole, there must be some $j \in E$ s.t.

$|f^{-1}(\{j\})| > 1$. 
But this is impossible, since $f$ is a bijection. Thus there is some $i : f(i) = \xi$ is even, so $T_1(i - f(i))$ is even, as desired. \[\square\]
Experiment:
1. Choose a bag A or B.
2. Choose a ball from the bag.

8. \( P(\text{blue})? \)

8. \( P(\text{bag A} | \text{blue})? \)

\[ S = \{ (A, \text{red}), (A, \text{blue}), (A, \text{green}), (B, \text{red}), (B, \text{blue}), (B, \text{green}) \} \]

Events:
- \{ \text{bag A} \}
- \{ \text{bag B} \}
- \{ \text{blue} \}
- \{ \text{green} \}
- \{ \text{red} \}

\( A \)
- 3 red
- 4 blue
- 1 green

\( B \)
- 1 red
- 10 blue
- 1 green
Given
\[ P(\text{bag A}) = \frac{1}{2}, \quad P(\text{bag B}) = \frac{1}{2} \]

\[
\begin{align*}
\quad P(\text{red} \mid \text{bag A}) &= \frac{3}{3+4+1} = \frac{3}{8} \\
\quad P(\text{blue} \mid \text{bag A}) &= \frac{4}{8} = \frac{1}{2} \\
\quad P(\text{green} \mid \text{bag A}) &= \frac{1}{8} \\
\quad P(\text{red} \mid \text{bag B}) &= \frac{1}{12} \\
\quad P(\text{blue} \mid \text{bag B}) &= \frac{10}{12} \\
\quad P(\text{green} \mid \text{bag B}) &= \frac{1}{12}
\end{align*}
\]
\( p(\text{blue}) = p(\text{blue} | \text{bag A}) \cdot p(\text{bag A}) \) \\
\( + p(\text{blue} | \text{bag B}) \cdot p(\text{bag B}) \) \\
(\text{law of total probability})

\( p(\text{bag A}) + p(\text{bag B}) = 1 \)

\( \{\text{bag A}\} \cap \{\text{bag B}\} = \emptyset \)

\[
= \frac{1}{2} \cdot \frac{1}{2} + \frac{10}{12} \cdot \frac{1}{2} = \frac{1}{4} + \frac{5}{12} \\
= \frac{8}{12} = \frac{3}{4}
\]

\text{Correction} \ 2/3
\( p(\text{bag A} | \text{blue}) = \frac{p(\text{blue} | \text{bag A}) \cdot p(\text{bag A})}{p(\text{blue})} \)

\[
\begin{align*}
= & \quad \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{4}} \\
= & \quad \frac{\frac{3}{16}}{\frac{3}{8}} \\
= & \quad \frac{3}{8}
\end{align*}
\]
6) Everyone in a class of 260 is assigned a random birthday in \( \mathbb{Z}_1, \ldots, 365 \). Call an (unordered) pair magical if they have the same birthday. What is the expected number of magical pairs?

\[
S = \left\{ (b_1, \ldots, b_{260}) : b_i \in \{1, \ldots, 365\} \right\}
\]

\[
= \{1, \ldots, 365\}^{260}
\]

\[
X(b_1, \ldots, b_{260}) = \# \text{ of magical pairs}
\]

\[
= \left| \left\{ (i, j) : b_i = b_j \text{ and } i \neq j \right\} \right|
\]

\[
\text{i,j} \in \{1, \ldots, 260\}
\]
\[ E[X] = \sum_{s \in S} p(s) X(s) \]

Idea: decompose as a sum of simple random variables

define for every \((i,j)\) s.t. \(1 \leq i < j \leq 260\)

\[ X_{ij}(b_1, \ldots, b_{260}) = X_{ij} = \begin{cases} 1 & \text{if } b_i = b_j \\ 0 & \text{otherwise} \end{cases} \]

Observe \( X = X_{12} + X_{13} + \cdots + X_{1,260} \)

\[ + X_{2,3} + X_{2,4} + \cdots + X_{2,260} \]

\[ = \sum_{i < j} X_{ij} \]
\[ \mathbb{E} X = \sum_{i<j} \mathbb{E} X_{ij} = p(\{b_i = b_j\}) \]

\[ = \frac{1}{\left| S \right|} \left| \left\{ b_i = b_j \right\} \right| \]

\[ = \frac{365^{259}}{365^{260}} = \frac{1}{365} \]

for every \( i<j \)
\[ EX = \left( \text{# of pairs } i < j \right) \times EX_{12} \]

\[ = \binom{260}{2} \frac{1}{365} = \frac{260 \cdot 259}{2 \cdot 365} \]

\[ = \frac{136 \cdot 259}{365} \frac{26}{73} = \frac{26 \cdot 259}{73} \]

\[ = 92.24 \]
False: Consider $S = \{H, T\}$

- $\mathbb{E}X = 0$
- $\mathbb{E}Y = 0$
- $\mathbb{V}(X) = \mathbb{E}X^2 - 1$
- $\mathbb{V}(Y) = \mathbb{E}Y^2 - 1$

Let $X = \{\pm 1\}$ if head

Let $Y = \{-1\}$ if tail.

But $X + Y = 0$

So $\mathbb{V}(X+Y) = 0$