Midterm 2 Solutions

(1) $f(1) = 1$, $f(k+1) = \sqrt{1+f(k)}$

$f(2) = \sqrt{1+1} = \sqrt{2}$, know is irrational.

Let $P(n)$ be the predicate "$f(n)$ is irrational".

<u>Basis Step</u>: $P(2)$ says: $\sqrt{2}$ is irrational, which is true.

<u>Inductive Step</u>: Assume $k \geq 2$. Assume $P(k)$ is true. Our Goal is to show $P(k+1)$.

Write: $\boxed{f(k+1) = \sqrt{1+f(k)}}$.

Assume for the sake of contradiction that $f(k+1)$ is <u>rational</u>.

Recall: $x \in \mathbb{R}$ is <u>rational</u> if $\exists\, p, q \in \mathbb{Z}$ s.t. $x = p/q$.

Thus, there exist integers $p, q \neq 0$ such that

$$f(k+1) = \frac{p}{q}. \quad \text{Therefore,} \quad \sqrt{1+f(k)} = p/q.$$

Squaring both sides, we get $1 + f(k) = \dfrac{p^2}{q^2}$,

which means $f(k) = \dfrac{p^2}{q^2} - 1 = \dfrac{p^2 - q^2}{q^2}$,

which is absurd since by the ind hyp. $f(k)$ is irrational. $\square$

② ① formulate conjecture.

<u>Observe</u>: \$2, \$4, \$6, ... any even amount.

\$5, \$10, \$15, ...

Can't make <u>3</u>, can't make \$1.

$7 - 5 = 2$. ✓

$9 - 5 = 4$ ✓

$11 - 5 = 6$ ✓

(odd # bigger than 5) $- 5 =$ positive even number!

<u>Conjecture</u>: Every positive integer amount except \$1 and \$3 can be made with \$2 and \$5 bills.

Proof: By strong Induction.

$P(n) =$ " \$n can be made from \$2 and \$5 bills "

Goal: $P(2), P(4), P(5), P(n)\ n \geq 6$.

Basis Step: Observe that $P(2), P(4)$ are true because 2, 4 are even, and $P(5)$ is trivially true.

Inductive Step: Assume $k \geq 5$. Assume $P(2), P(4), P(5), \ldots P(k)$ hold.

Goal: $P(k+1)$ is true.

Observe: $k+1 = 2 + (k-1)$ and $k-1 \geq 4$.

By the inductive hypothesis, $P(k-1)$ holds.

So there is a way to write \$k-1 in \$2, \$5 bills.

Adding a \$2 bill, we can make \$k+1.

This completes the induction. $\square$.

③ How many poker hands contain one ace, no king, ≥ 1 heart?

By sum rule,

$$\#\text{hands} = \underbrace{\#\text{hands with one ace, no king}}_{A} - \underbrace{\#\text{hands with one ace, no king, no hearts.}}_{B}$$

<u>A:</u> Consider the process:

	Ways
① Choose one ace	4
② Choose 4 more cards, no kings or aces.	$\binom{44}{4}$

By product rule, $A = 4 \cdot \binom{44}{4}$.

B Process: ① Choose one ace $\quad 3$

② Choose 4 cards, $\quad \binom{33}{4}$

no king, ace, heart

$52 - 13$ hearts $= 39$

$39 - 3$ kings $- 3$ ace

$= 33$ cards

Product rule: $3 \cdot \binom{33}{4}$.

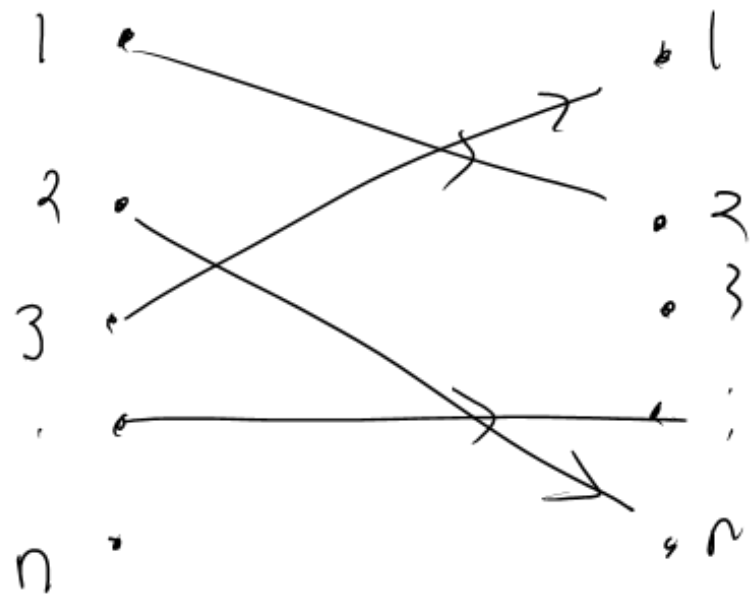
Answer $= 4 \cdot \binom{44}{4} - 3 \cdot \binom{33}{4}$.

(4) Given n <u>odd</u> positive, $f : \{1, \ldots n\} \to \{1 \ldots n\}$.

<u>Goal</u>: $(1-f(1))(2-f(2)) \cdots (n-f(n))$ is <u>even</u>.

<u>Observe</u>: Sufficient to show there exists some i s.t $(i-f(i))$ is even.

<u>Observe</u>: If i and $f(i)$ are both odd, then $i-f(i)$ is even!



Let $O = \{k : 1 \leq k \leq n \text{ and } k \text{ is odd}\}$.

<u>Claim</u>: There exists some $i \in O$ such that $f(i) \in O$.

<u>Proof</u>: By contradiction. Assume not, i.e.

$$f(O) \subseteq \{k : k \text{ is even}\} = E$$

Observe that $|O| = |E| + 1$, since n is odd. Thus, f maps $|E|+1$ elements to $|E|$ elements. By pigeonhole, there must be some $j \in E$ s.t.

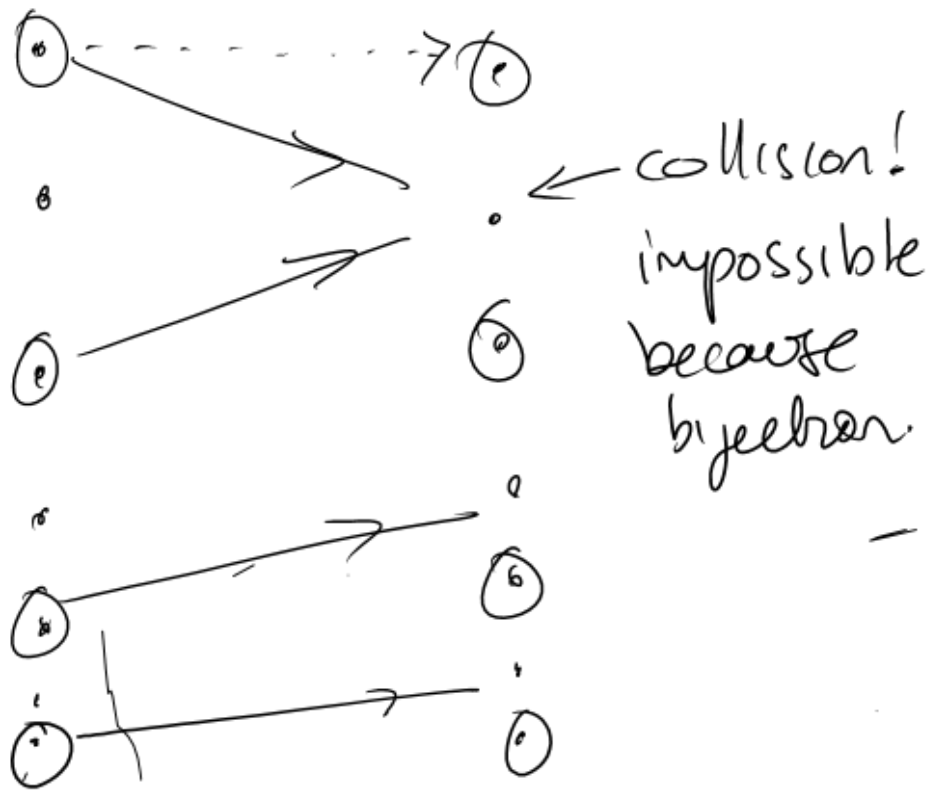
$$|f^{-1}(\{j\})| > 1.$$

But this is impossible, since f is a bijection.

Thus there is some i : $f(i)-i$ is even,

so $\prod(i-f(i))$ is even, as desired.

$\square$



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Experiment:

1. Choose a bag A or B.
2. Choose a ball from the bag

Q) $p(\text{blue})$?

Q) $p(\text{bag A} \mid \text{blue})$?

A	B
3 red	1 red
4 blue	10 blue
1 green	1 green

$$S = \{ (A, \text{red}), (A, \text{blue}), (A, \text{green}), (B, \text{red}), (B, \text{blue}), (B, \text{green}) \}.$$

Events: $\{\text{bag A}\}$ $\{\text{bag B}\}$

$\{\text{blue}\}$ $\{\text{green}\}$ $\{\text{red}\}$

Given $P(\text{bag A}) = 1/2$, $P(\text{bag B}) = 1/2$.

$$\begin{cases} P(\text{red} \mid \text{bag A}) = \dfrac{3}{3+4+1} = \dfrac{3}{8} \\ P(\text{blue} \mid \text{bag A}) = \dfrac{4}{8} = 1/2 \\ P(\text{green} \mid \text{bag A}) = \dfrac{1}{8} \end{cases}$$

$$\begin{cases} P(\text{red} \mid \text{bag B}) = \dfrac{1}{12} \\ P(\text{blue} \mid \text{bag B}) = \dfrac{10}{12} \\ P(\text{green} \mid \text{bag B}) = \dfrac{1}{12} \end{cases}$$

Q1) $p(\text{blue}) = p(\text{blue} \mid \text{bag A})\, p(\text{bag A}) + p(\text{blue} \mid \text{bag B})\, p(\text{bag B})$

(law of total probability,

$p(\text{bag A}) + p(\text{bag B}) = 1$

$\{\text{bag A}\} \cap \{\text{bag B}\} = \phi$)

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{10}{12} \cdot \frac{1}{2} = \frac{1}{4} + \frac{5}{12}$$

$$= \frac{8}{12} = \frac{3}{4}$$

Correction 2/3

Q2) $P(\text{bag A} \mid \text{blue}) \underset{\text{Bayes}}{=} \dfrac{P(\text{blue} \mid \text{bag A})\, P(\text{bag A})}{P(\text{blue})}$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{4}} = \cancel{\frac{3}{16}} \quad \frac{3}{8}$$

$2/3$

(6) Everyone in a class of 260 is assigned a random birthday in $\{1, \ldots, 365\}$. Call an (unordered) pair <u>magical</u> if they have the <u>same birthday</u>. What is the <u>expected #</u> of <u>magical pairs</u>?

$$S = \{ (b_1, \ldots, b_{260}) : b_i \in \{1, \ldots, 365\} \}$$

$$= \{1, \ldots 365\}^{260}$$

$$X(b_1, \ldots b_{260}) = \# \text{ of magical pairs}$$

$$= \left| \{ (i,j) : b_i = b_j \text{ and } i < j \} \right|$$

$$i, j \in \{1, \ldots 260\}$$

$$\mathbb{E}X = \sum_{s \in S} p(s)\, X(s).$$

<u>Idea</u>: decompose as a sum of simple random variables.

define for every (i,j) s.t. $1 \le i < j \le 260$

$$X_{ij}(b_1, \dots b_{260}) = X_{ij} = \begin{cases} 1 & \text{if } b_i = b_j \\ 0 & \text{otherwise} \end{cases}$$

Observe $X = X_{1,2} + X_{1,3} + \cdots X_{1,260}$

$$+ X_{2,3} + X_{2,4} + \cdots X_{2,260}$$

$$\cdots$$

$$= \sum_{i<j} X_{ij}$$

$$\mathbb{E}X = \sum_{i<j} \mathbb{E}X_{ij}.$$

$\{(b_1 \ldots b_{260}) : b_i = b_j\}$

<u>Calculate:</u>

$$\mathbb{E}X_{ij} = P(\{b_i = b_j\})$$

$$= \frac{|\{b_i = b_j\}|}{|S|}$$

$$= \frac{365^{259}}{365^{260}} = \underline{\underline{\frac{1}{365}}}.$$

for every $i<j$.

$$\mathbb{E}X = \left(\#\text{ of pairs } i<j \right) \times \mathbb{E}X_{12}$$

$$= \binom{260}{2} \frac{1}{365} = \frac{260 \cdot 259}{2 \cdot 365}$$

$$= \frac{\overset{26}{\cancel{130}} \cdot 259}{\cancel{365}\ 73}$$

$$= \frac{26 \cdot 259}{73}$$

$$= 92.24$$

(7) T or F: $V(X+Y) \geq V(X)+V(Y)$.

<u>False</u>: Consider $S=\{H,T\}$

$\mathbb{E}X=0$

$$X = \begin{cases} +1 & \text{if head} \\ -1 & \text{if tail} \end{cases}$$

$\mathbb{E}Y=0$

$$Y = \begin{cases} -1 & \text{if head} \\ +1 & \text{if tail.} \end{cases}$$

$V(X) = \mathbb{E}X^2 = 1$

$V(Y) = \mathbb{E}Y^2 = 1$

but $X+Y=0$

So $V(X+Y)=0$