

Math 55 Spring 2016 Practice Midterm 2

Nikhil Srivastava

80 minutes, closed book, one page of notes allowed

1. (8 points) Consider the function recursively defined by:

$$f(1) = 1; \quad f(k+1) = \sqrt{1 + f(k)} \quad \text{when } k \geq 1.$$

Given that $f(2) = \sqrt{2}$ is irrational, prove, using induction, that $f(n)$ is irrational for all integers $n \geq 2$.

2. (8 points) Which amounts of money can be made using an unlimited supply of \$2 and \$5 bills? Formulate a conjecture and prove it.
3. (8 points) How many poker hands (of 5 cards, from a deck of 52) contain exactly one ace, no king, and at least one heart?
4. (8 points) Suppose n is an odd positive integer and $f : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is a bijection. Show that the number

$$\prod_{i=1}^n (i - f(i)) = (1 - f(1)) \cdot (2 - f(2)) \cdot \dots \cdot (n - f(n))$$

must be even. (hint: pigeonhole)

5. (8 points) Suppose Bag A contains 3 red balls, 4 blue balls, and 1 green ball, and Bag B contains 1 red ball, 10 blue balls, and 1 green ball. Consider the experiment where I choose a bag uniformly at random (with equal probability) and choose a ball from that bag uniformly at random. (a) What is the probability that I draw a blue ball? (b) Given that I draw a blue ball, what is the probability that I chose Bag A?
6. (8 points) Suppose that everyone in a class of 260 students is born on a uniformly random day of the year, chosen from $\{1, \dots, 365\}$. Let us call an (unordered) pair of students *magical* if both have the same birthday. What is the expected number of magical pairs in the class? (hint: linearity of expectation)
7. (2 points) True or false: If X and Y are random variables, then $V(X + Y) \geq V(X) + V(Y)$, where V is the variance.