2.3, 28 If an inverse \( g \) existed, then we would have \( f \circ g(x) = x \). Take \( x = -1 \); then \( f(g(-1)) = e^{g(-1)} \geq 0 \), so \( f(g(-1)) \neq -1 \). The problem here is that \( f \) is not surjective as a function to the real numbers. If the codomain is restricted to the positive real numbers, then \( g(x) = \ln(x) \) is an inverse. We couldn’t do this before, because \( \ln(x) \) is not well-defined on all reals, but it is well-defined as a function \( g: \mathbb{R}^{\geq 0} \to \mathbb{R} \).

2.3, 34 Yes. We will prove the contrapositive. Suppose that \( x \neq y \) and \( g(x) = g(y) \). Then, \((f \circ g)(x) = (f \circ g)(y)\), so \( f \circ g \) is also not one-to-one.

2.3, 40 (a) First, we show that \( f(S \cup T) \subset f(S) \cup f(T) \). Suppose \( x \in f(S \cup T) \). This means there is either a \( s \in S \) such that \( x = f(s) \), or a \( t \in T \) such that \( x = f(t) \). In the former case, this means that \( x \in f(S) \). In the latter case, this means that \( x \in f(T) \). Thus, in either case, \( x \in f(S) \cup f(T) \). Now, we will show that \( f(S) \cup f(T) \subset f(S \cup T) \). Suppose that \( y \in f(S) \cup f(T) \). This means that either there is \( s \in S \) such that \( y = f(s) \) (i.e. \( y \in f(S) \)) or there is \( t \in T \) such that \( y = f(t) \). In either case, \( y \in f(S \cup T) \).

(b) We will show that \( f(S \cap T) \subset f(S) \cap f(T) \). Suppose that \( x \in f(S \cap T) \). Then there is a \( r \) such that \( r \in S \) and \( r \in T \) and \( x = f(r) \). So, \( x \in f(S) \) and \( x \in f(T) \). Note that it is not true that \( f(S) \cap f(T) \subset f(S \cap T) \). A counterexample is: take \( f(x) = x^2 \), and take \( S = \{-1\} \) and \( T = \{1\} \).

2.3, 44 (a) First, we will show that \( f^{-1}(S \cup T) \subset f^{-1}(S) \cup f^{-1}(T) \). Suppose that \( x \in f^{-1}(S \cup T) \). This means \( f(x) \in S \cup T \). So, \( f(x) \in S \) or \( f(x) \in T \). So, \( x \in f^{-1}(S) \) or \( f^{-1}(T) \), i.e. \( x \in f^{-1}(S) \cup f^{-1}(T) \). Now, we will show that \( f^{-1}(S) \cup f^{-1}(T) \subset f^{-1}(S \cup T) \). Suppose that \( y \in f^{-1}(S) \cup f^{-1}(T) \). This means either \( f(y) \in S \) or \( f(y) \in T \), i.e. \( f(y) \in S \cup T \). Thus, \( y \in f^{-1}(S \cup T) \).

(b) First, we will show that \( f^{-1}(S \cap T) \subset f^{-1}(S) \cap f^{-1}(T) \). Suppose that \( x \in f^{-1}(S \cap T) \). This means that \( f(x) \in S \) and \( f(x) \in T \). So, \( x \in f^{-1}(S) \) and \( f^{-1}(T) \), i.e. \( x \in f^{-1}(S) \cap f^{-1}(T) \). Now, we will show that \( f^{-1}(S) \cap f^{-1}(T) \subset f^{-1}(S \cap T) \). Suppose that \( y \in f^{-1}(S) \cap f^{-1}(T) \). This means either \( f(y) \in S \) or \( f(y) \in T \), i.e. \( f(y) \in S \cap T \). Thus, \( y \in f^{-1}(S \cap T) \).\(^1\)

2.4, 26ace (a) The rule is \( a_{n+1} = a_n + (2n + 1) \) (assuming the sequence starts at 1). The next three elements of the sequence are 123, 146, 171.

(c) The rule is \( a_{n+1} = a_n + 1 \), where the expressions are in base 2 (binary). The next three elements in the sequence are 1100, 1101, 1110.

(e) The rule is \( a_{n+1} = a_n + 2 \cdot 3^{n-1} \). The next three elements in the sequence are 59048, 177146, 531440.

2.4, 32 (a) The summands are 2 if \( j \) is odd and 0 if \( j \) is even. Thus, the value is 8. (b) This sum can be written \( \sum_{j=0}^{8} 3^j - \sum_{k=0}^{8} 2^j = \frac{3^9-1}{3-1} - \frac{2^9-1}{2-1} = 9330 \). (c) This sum can be written \( 2 \sum_{j=0}^{8} 3^j + 3 \sum_{k=0}^{8} 2^j = 2 \cdot \frac{3^9-1}{3-1} + 3 \cdot \frac{2^9-1}{2-1} = 21215 \). (d) The summand can be written \( 2^{j+1} - 2^j = 2^j \). Thus, the sum is \( 2^8 - 1 = 511 \).

\(^1\)Note that I literally copied the proof from part (a) and replaced \( \cup \) with \( \cap \), and replaced “or” with “and.” Of course, you can’t always do this.
2.4, 34 (a) We can separate the sum and write it as \( \sum_{i=1}^{3} \sum_{j=1}^{2} i - \sum_{i=1}^{3} \sum_{j=1}^{2} j \). (b) We can separate the sum and write it as \( 2 \sum_{i=1}^{3} \sum_{j=1}^{2} i + 3 \sum_{i=1}^{3} \sum_{j=1}^{2} j = 2 \cdot 6 - 3 \cdot 3 = 3 \). (c) Since \( i \) is not involved in the summand, we can write this sum \( 3 \sum_{j=0}^{2} j = 3 \cdot 3 = 9 \). (d) We can rewrite this sum as \((\sum_{i=0}^{3} i^2)(\sum_{j=0}^{3} j^3) = (1+4)(1+8+27) = 5(36) = 180\). 

2.4, 46 This product is 0!1!2!3!4! = 1 \cdot 1 \cdot 2 \cdot 6 \cdot 24 = 288. Another way is to notice that 0!1!2!3!4! = 2^3 3^2 4 = 8 \cdot 9 \cdot 4 = 288.

2.5, 2 (a) Countably infinite. \( f(n) = n + 10 \). This is well-defined since if \( n > 0 \) then \( f(n) > 10 \). Its inverse is given by \( g(m) = m - 10 \). One can check that \( g(f(n)) = n \) and \( f(g(m)) = m \). (b) Countably infinite. \( f(n) = -2n + 1 \), with inverse \( g(m) = \frac{1-m}{2} \). (c) Finite. (d) Uncountable. (e) Countably infinite. \( f(n) = (2, \frac{n}{2}) \) if \( n \) is even, and \( f(n) = (3, \frac{n+1}{2}) \) if \( n \) is odd. The inverse is \( g(2,b) = 2b \) and \( g(3,b) = 2b - 1 \). (f) Countably infinite. \( f(n) = 10 \frac{n}{2} \) if \( n \) is even, and \( f(n) = -10 \frac{n-1}{2} + 1 \) if \( n \) is odd. The inverse is given by \( g(m) = \frac{m}{3} \) is \( m \) is positive, and \( g(m) = \frac{m}{3} + 1 \) if \( m \) is nonpositive.

2.5, 10 (a) \( A = [0,1] \cup \{3\} \) and \( B = [0,1] \). (b) \( A = [0,1] \cup \mathbb{Z} \) and \( B = [0,1] \). (c) \( A = [0,1] \) and \( B = [0,\frac{1}{2}] \).

2.5, 20 \(|A| = |B| \) implies there is a bijection \( f : A \to B \). \(|B| = |C| \) implies there is a bijection \( g : B \to C \). Then \( g \circ f : A \to C \) is a bijection, so \(|A| = |C| \).

2.5, 40 Suppose we had \( f : S \to \mathcal{P}(S) \) which is surjective (onto). Define \( T = \{ s \in S \mid s \not\in f(s) \} \subset S \). Since \( f \) is onto, there is a \( t \) such that \( f(t) = T \). There are two possibilities: \( t \in f(t) = T \) or \( t \not\in f(t) = T \). The former cannot occur, since this would imply that \( t \not\in f(t) = T \) by definition of \( T \). The latter cannot occur, since this would imply that \( t \in T = f(t) \). Thus we have a contradiction.

4.1, 16 If \( a \equiv b \pmod{m} \), then \( m \mid a - b \). So, there is an integer \( k \) such that \( a - b = mk \), i.e. \( a = b + mk \). Taking \( \text{mod } m \), we find that \( a \mod m = b \mod m \).

4.1, 24 (a) \(-3 \) (b) \(-12 \) (c) 94

4.1, 30 (a) 13 (b) 19

4.1, 37 (a) Take \( m = 4, c = 2, a = 1 \) and \( b = 3 \). (b) Take \( m = 4, a = b = 2 \), and \( c = 1 \) and \( d = 5 \).

4.1, 38 There are four possibilities: \( n \equiv 0,1,2,3 \). If \( n \equiv 0,2 \), then \( n^2 \equiv 0 \). If \( n \equiv 1,3 \), then \( n^2 \equiv 1 \).

4.1, 39 Let \( m = 4k + 3 \). Then \( m \equiv 3 \pmod{4} \). If \( m = a^2 + b^2 \), then modulo 4, \( m \) can only be \( 0 + 0, 0 + 1, 1 + 1 \). None of these are equivalent to 3 modulo 4.

4.2, 2 (a) 101000001 (b) 1111111111 (c) 11000100100011000

4.2, 4 (a) 27 (b) 693 (c) 958 (d) 31775

4.2, 28 (a) First, we can simplify \( 123 \equiv 22 \pmod{101} \). Next, we write 1001 in binary as 1111101001. Taking powers, we have \( 22^2 \equiv 80 \pmod{101}, 22^3 \equiv 43 \pmod{101}, 22^4 \equiv 37 \pmod{101}, 22^5 \equiv 6 \pmod{101}, 22^6 \equiv 31 \pmod{101}, 22^7 \equiv 76 \pmod{101}, 22^8 \equiv 56 \pmod{101}, 22^9 \equiv 20 \pmod{101} \). Using this, we take \( 22 \cdot 43 \cdot 6 \cdot 31 \cdot 76 \cdot 56 \cdot 20 \equiv 82 \pmod{101} \).
4.2, 32 Let $a_r \cdots a_0$ be the decimal representation of the number $n$. We can write this as $n = a_0 + 10a_1 + 100a_2 + \cdots$. Note that $10 \equiv -1 \pmod{11}$, $10^2 \equiv 1 \pmod{11}$. Continuing, we find that $10^k \equiv (-1)^k \pmod{11}$. Thus, we have that $n \equiv a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^r a_r \pmod{11}$. Further, $n \equiv 0 \pmod{11}$ if and only if $n$ is divisible by 11. So, moving all negative terms to one side, $n \equiv 0 \pmod{11}$ becomes $a_0 + a_2 + \cdots \equiv a_1 + a_3 + \cdots \pmod{11}$ as desired.