Homework 14 Out of Textbook Problems Solutions

1. Assume $G = (V, E)$ is a connected multigraph with $n \geq 2$ vertices. Let $S \subseteq V$ be the set of vertices with odd degree. If $S = \emptyset$ then by the Euler circuit theorem $G$ has an Euler circuit, and we are done. Otherwise, since the sum of the degrees of the vertices is even by the handshaking theorem, we must have $|S| = k$ for some even $k$. Order the vertices in $S$ arbitrarily as $v_1, v_2, \ldots, v_k$.

Add edges
\[ \{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{k-1}, v_k\} \]

to $G$, call the resulting graph
\[ G' = (V, E \cup \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{k-1}, v_k\}\}) \].

Note that for every $v \in S$, the degree of $v$ in $G'$ is one more than the degree of $v$ in $G$, so in particular it is even. Since we have not added any edges incident to the other vertices $V \setminus S$, their degrees are the same as they were in $G$, which were in particular even. Thus, the degrees of all vertices in $G'$ are even. Since $G'$ is also connected and has at least 2 vertices, it must have an Euler circuit by the Euler circuit theorem.

The total number of edges we have added is exactly $k/2$ (since $k$ is even). When $n$ is even, this is at most $n/2 = \lfloor n/2 \rfloor$. When $n$ is odd, we must have $k \leq n - 1$, so again $k/2 \leq \frac{n-1}{2} = \lfloor n/2 \rfloor$, as desired.

2. ($\Rightarrow$). Assume $G = (V, E)$ is a simple graph and $c : V \to \{1, \ldots, k\}$ is a $k$–coloring of $G$, i.e., for every $uv \in E$ we have $c(u) \neq c(v)$. Define the sets:
\[ V_i = \{v \in V : c(v) = 1\}, V_2 = \{v \in V : c(v) = 2\}, \ldots, V_k = \{v \in V : c(v) = k\}. \]

Note that $V_i \cap V_j = \emptyset$ whenever $i \neq j$, and moreover $V = V_1 \cup \ldots \cup V_k$, since every vertex is assigned a color by $c$.

Assume $i = 1, \ldots, k$ is arbitrary and let $u, v$ be arbitrary vertices in $V_i$. Since $c(u) = c(v)$, we know by the definition of a coloring that $uv \notin E$. Thus, no two vertices in $V_i$ are adjacent in $G$, as desired.

($\Leftarrow$). Assume $G = (V, E)$ is a simple graph and $V_1, \ldots, V_k$ is a partition of its vertices such that for every $i$, no two vertices in $V_i$ are adjacent in $G$. We will produce a $k$–coloring of $G$. Consider the function $c : V \to \{1, \ldots, k\}$ defined by:
\[ c(v) = i \quad \text{s.t.} \quad v \in V_i. \]

Since $V_1, \ldots, V_k$ is a partition of $V$, every $v \in V$ appears in exactly one set in the partition, so $c$ is a well-defined function. Assume now that $uv \in E$ is an arbitrary edge of $G$. Since no two vertices in any $V_i$ are adjacent, we must have $u \in V_i$ and $v \in V_j$ for some $i \neq j$. But now $c(u) \neq c(v)$, so indeed $c$ is a $k$–coloring of $G$, as desired.
3. Assume $G = (V, E)$ is an arbitrary simple graph that is not connected. Let $G_1, \ldots, G_k$ be the connected components of $G$, and note that $k > 1$ since if $k = 1$, $G$ would be connected. Note that every vertex in $G$ lies in exactly one connected component.

We will show that the complement $G' = (V, \{\{u, v\} : \{u, v\} \not\in E\})$ is connected. The proof is based on the following observation: for all pairs of vertices $u, v \in V$ such that $u$ and $v$ are in distinct connected components of $G$, $u$ cannot be adjacent to $v$ in $G$.

To see why, assume for contradiction that there exist $u \in G_i, v \in G_j$ with $i \neq j$ and $uv \in E$. Then we could add the vertex $v$ and the edge $uv$ to $G_i$ to obtain a strictly larger connected subgraph of $G$, contradicting the maximality of $G_i$ (which is part of the definition of a connected component). Thus, we conclude that whenever $u$ and $v$ are in distinct connected components of $G$, they must be adjacent in the complement $G'$.

We now show that $G'$ is connected. Let $u$ and $v$ be arbitrary vertices in $G'$. There are two cases:

**Case 1:** $u$ and $v$ are in distinct connected components $G_i$ and $G_j$ with $i \neq j$. By the observation, $uv$ is an edge of $G'$, and in particular there is a path from $u$ to $v$ in $G'$ (the path consisting of the single edge $uv$.)

**Case 2:** There exists an $i$ such that $u \in G_i$ and $v \in G_i$. Choose any other connected component $G_j$ such that $j \neq i$ (this is possible since there are at least two connected components). Let $w$ be an arbitrary vertex in $G_j$. By the observation, the edges $uw$ and $vw$ must be present in $G'$. Thus, there is a path of length 2, namely $uw, vw$ from $u$ to $v$ in $G'$.

Thus, we have shown that an arbitrary pair of vertices in $G'$ is connected by a path in $G'$, so $G'$ is connected, as desired.