8.4.18 The number of ways to choose 14 balls is the coefficient of $x^{14}$ in the generating function
\[
(1 + x + x^2 + \cdots + x^{100})(1 + x + x^2 + \cdots + x^{100})(x^3 + x^4 + \cdots + x^{10})
\]

because each term in the product is of the form $x^i x^j x^k$ where $0 \leq i \leq 100$, $0 \leq j \leq 100$, and $3 \leq k \leq 10$, and therefore each copy of $x^{14}$ in the product corresponds to such a trio $i, j, k$ such that $i + j + k = 14$ which in turn corresponds to a choice of some number of red, green, and blue balls. The coefficient of $x^{14}$ in this product is 68.

8.4.19 See back of the book.

8.4.30:

(a) $2G(x)$. The generating function given is
\[
\sum_{n=0}^{\infty} 2a_n x^n = 2 \sum_{n=0}^{\infty} a_n x^n = 2G(x)
\]

(b) $x^2G(x) - a_1 x^3 - a_0 x^2$. The generating function given is
\[
0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + a_2 \cdot x^4 + a_3 \cdot x^5 + \cdots = \sum_{n=2}^{\infty} a_n x^{n+2} = x^2 \sum_{n=2}^{\infty} a_n x^n = x^2(G(x) - a_1 x - a_0)
\]

(e) $G'(x)$. The derivative of the generating function $G(x)$ is given by differentiating term by term. Since
\[
G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots
\]
we have
\[
G'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n
\]
is the generating function corresponding to the sequence given.

8.4.39 See back of the book.

8.4.41 See back of the book.

8.4.43 See back of the book.

8.5.14 Let $F$ be the set of permutations containing the word fish, $R$ be the set of permutations containing the word rat, and $B$ be the set of permutations containing the word bird. Then the number of permutations not containing any of those three words is $26! - |F| - |R| - |B| + |F \cap R| + |R \cap B| + |B \cap F| - |F \cap R \cap B|$. Note that due to the shared letters $i$ and $r$, $R \cap B = B \cap F = F \cap R \cap B = \emptyset$. So we are trying to compute $26! - |F| - |R| - |B| + |F \cap R|$. There are 23 places for the word fish to be placed within the permutation and 22! ways to permute the remaining letters, so $|F \cap R| = 23 \cdot 22! = 23!$. Similarly, $|R| = 24 \cdot 23! = 24!$.

Now consider how many ways to place fish and rat in a line of 26 letters. If fish is placed at place $i$ with $4 \leq i \leq 20$ then there are $(i - 2) + (22 - i - 2)$ ways of placing the word rat (either before or after the word fish). If fish is placed at place $1$ or $23$ then there are $22 - 2$ ways of placing rat, if fish is placed at place $2$ or $22$ then there are $21 - 2$ ways of placing rat, and if fish is placed at place $3$ or $21$ then there are $20 - 2$ ways of placing rat. Thus the total number of ways of placing the two words is $2 \cdot 20 + 2 \cdot 19 + 2 \cdot 18 + 17 \cdot 18$. Now for each of these placements there are 19! ways of placing the remaining letters. So $26! - |F| - |R| - |B| + |F \cap R| = 26! - 23! - 24! + 420 \cdot 19!$. 

8.5.22 We want to prove:

\[ |A_1 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_n| \]

Base case: \( n = 2 \). Proved already in lecture/book.

Inductive step: assume

\[ |A_1 \cup \cdots \cup A_{n-1}| = \sum_{1 \leq i \leq n-1} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots + (-1)^n|A_1 \cap \cdots \cap A_{n-1}| \]

We know

\[ |A_1 \cup \cdots \cup A_n| = |A_1 \cup \cdots \cup A_{n-1}| + |A_n| - |(A_1 \cup \cdots \cup A_{n-1}) \cap A_n| \]

Thus

\[
|A_1 \cup \cdots \cup A_{n-1}| = \sum_{1 \leq i \leq n-1} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots + (-1)^n|A_1 \cap \cdots \cap A_{n-1}|
\]

\[ + |A_n| - |(A_1 \cup \cdots \cup A_{n-1}) \cap A_n| \]

\[ = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots + (-1)^n|A_1 \cap \cdots \cap A_{n-1}| \]

\[ - |(A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)| \]

Using the induction hypothesis again, we get

\[
|(A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)| = \sum_{1 \leq i \leq n} |A_i \cap A_n| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j \cap A_n| + \cdots + (-1)^n|A_1 \cap \cdots \cap A_{n-1} \cap A_n| \]

Now we have

\[
|A_1 \cup \cdots \cup A_{n-1}| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots + (-1)^n|A_1 \cap \cdots \cap A_{n-1}|
\]

\[ - |(A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)| \]

\[ = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots + (-1)^n|A_1 \cap \cdots \cap A_{n-1}|
\]

\[ - \sum_{1 \leq i \leq n-1} |A_i \cap A_n| + \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j \cap A_n| - \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_{n-1} \cap A_n| \]

\[ = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_n| \]

as desired.

8.5.24 Let \( A \) be the event that tails comes up exactly three times, \( B \) the event that the first and last flips come up tails, and \( C \) the event that the second and fourth flips come up heads. Then \( |A| = \binom{5}{3} = 10 \), \( |B| = 2^3 \), \( |C| = 2^3 \), \( |A \cap B| = 3 \), \( |A \cap C| = 3 \) since \( A \cap B = \{TTHHT, THTHT, THHTT\} \), \( |B \cap C| = 2 \) since \( B \cap C = \{THTHT, THHHT\} \), and \( |C \cap A| = |A \cap B \cap C| = 1 \) since \( C \cap A = A \cap B \cap C = \{THTHT\} \). By inclusion-exclusion, \( |A \cup B \cup C| = 10 + 8 + 8 - 3 - 2 - 1 + 1 = 21 \). There are \( 2^5 = 32 \) total outcomes so the probability of at least one of these events happening is \( 21/32 \).

8.5.26 Let \( E_1, E_2, E_3, \) and \( E_4 \) be events in a sample space \( S \). Then

\[
p(E_1 \cup E_2 \cup E_3 \cup E_4) = \frac{|E_1 \cup E_2 \cup E_3 \cup E_4|}{|S|}
\]

\[ = \frac{|E_1| + |E_2| + |E_3| + |E_4| - |E_1 \cap E_2| - |E_1 \cap E_3| - |E_1 \cap E_4| - |E_2 \cap E_3| - |E_2 \cap E_4| - |E_3 \cap E_4|}{|S|}
\]

\[ = \sum_i p(E_i) - \sum_{i<j} p(E_i \cap E_j) \]
since no three events can occur at the same time.

8.6.2  \[1000 - 450 - 622 - 30 + 111 + 14 + 18 - 9 = 32\]

8.6.8  By theorem 1, the number of onto functions is

\[5^7 - 5 \cdot 4^7 + \left(\frac{5}{2}\right) \cdot 3^7 - \left(\frac{5}{3}\right) \cdot 2^7 + \left(\frac{5}{4}\right) \cdot 1^7\]

which is 16800.

8.6.14  By theorem 2, the number of ways for the hats to be handed back such that nobody receives his own hat is

\[10! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\right)\]

and the total number of ways the hats can be handed back is 10!. So the probability is

\[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\]

which comes to 16481/44800 ≈ 0.3678

8.6.16  \(D_n\). This is because we can think of the seat each student is assigned for the first class as “belonging” to that student, and every assignment of seats for the second class is a way of giving a seat to each student such that none of the students is given “his own” seat.

8.6.18  Consider person 1. In any derangement, he may get any one of \(n-1\) hats back (not his own). Let’s say he gets hat \(i\). Then either person \(i\) gets hat 1 or person \(i\) doesn’t get hat 1. The number of derangements where person 1 gets hat \(i\) and person \(i\) gets hat 1 is exactly \(D_{n-2}\), the number of ways the remaining \(n-2\) people can have their hats returned. The number of derangements where person 1 gets hat \(i\) and person \(i\) doesn’t get hat 1 is exactly \(D_{n-1}\) because it is the number of ways that persons 2, 3, . . . , \(n\) can be given hats 1, 2, 3, . . . , \(i-1, i+1, \ldots, n\) such that no person gets his hat and person \(i\) doesn’t get hat 1 which is exactly the same as the number of ways persons 2, 3, . . . , \(n\) can be given hats 2, 3, . . . , \(n\) such that no person gets his hat. Thus the total number of derangements of \(n\) elements is \((n-1)(D_{n-1} + D_{n-2})\).

9.1.6  

(a)  not reflexive: if \(x \neq 0\) then \(x + x = 2x \neq 0\) so \((x, x) \notin R\).

symmetric: if \(x+y=0\) then \(y+x=0\).

not antisymmetric: for any \(x \neq 0\), \((x, -x) \in R\) and \((-x, x) \in R\) but \(x \neq -x\).

not transitive: for any \(x \neq 0\), \((x, -x) \in R\) and \((-x, x) \in R\) but \((x, x) \notin R\).

(d)  not reflexive: if \(x \neq 0\) then \(x \neq 2x\) so \((x, x) \notin R\).

not symmetric: for example \((2, 1) \in R\) since \(2 = 2 \cdot 1\) but \((1, 2) \notin R\) since \(1 \neq 2 \cdot 2\).

antisymmetric: if \((x, y) \in R\) and \((y, x) \in R\) then \(x = 2y\) and \(y = 2x\) so \(x = 4x\), which can only be true if \(x = 0\).

Since \(y = 2x\), we must have \(y = 0\) as well, so \(y = x\).

not transitive: for example \((4, 2) \in R\) and \((2, 1) \in R\) but \((4, 1) \notin R\) as \(4 \neq 2 \cdot 1\).

(h)  not reflexive: for example, \((2, 2) \notin R\) since \(2 \neq 1\).

symmetric: if \((x, y) \in R\) then \(x = 1\) or \(y = 1\) so \((y, x) \in R\).

not antisymmetric: for example \((1, 0) \in R\) and \((0, 1) \in R\) but \(1 \neq 0\).

not transitive: for example \((0, 1) \in R\) and \((1, 2) \in R\) but \((0, 2) \notin R\)

9.1.10
(a) Let $S$ be the set $\{a\}$ and $R$ the relation $\{(a,a)\}$.

(b) Let $S$ be the set $\{a, b, c\}$ and $R$ the relation $\{(a, b), (b, a), (a, c)\}$. Then $R$ is not symmetric since $(a, c) \in R$ but $(c, a) \notin R$ and $R$ is not antisymmetric since $(a, b) \in R$ and $(b, a) \in R$ but $a \neq b$.

9.1.47 See back of the book

9.1.49 See back of the book

9.3.2b
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

9.3.4b $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$

9.3.19b See back of the book

9.3.24 $\{(a, a), (b, b), (c, c), (b, a), (a, c), (b, c)\}$

9.3.32 The relation in graph 26 is reflexive, not irreflexive, not symmetric, not antisymmetric, not asymmetric, not transitive.

The relation in graph 27 is not reflexive, not irreflexive, symmetric, not antisymmetric, not asymmetric, not transitive.

The relation in graph 28 is reflexive, not irreflexive, symmetric, not antisymmetric, not asymmetric, transitive

9.5.2

(b) equivalence relation: each person has the same parents as himself/herself, if $a$ and $b$ have the same parents then $b$ and $a$ have the same parents, if $a$ has the same parents as $b$ and $b$ has the same parents as $c$ then $a$ has the same parents as $c$.

(c) not an equivalence relation: reflexive and symmetric but not transitive as it may be that $a$ and $b$ share a mother and $b$ and $c$ share a father but $a$ and $c$ have no parents in common.

(d) not an equivalence relation: reflexive and symmetric but not transitive as it may be that $a$ and $b$ have met and $b$ and $c$ have met but $a$ and $c$ have not met.

9.5.3 See back of the book

9.5.10 Let $B$ be the set of equivalence classes of $A$ and $f : A \to B$ the function that takes an element of $A$ to its equivalence class. If $(x, y) \in R$ then $x$ and $y$ are in the same equivalence class so $f(x) = f(y)$. Conversely, if $f(x) = f(y)$ then $x$ and $y$ are in the same equivalence class so $(x, y) \in R$.

9.5.16 Reflexivity: $((a, b), (a, b)) \in R$ for all $a, b \in \mathbb{R}$ because $ab = ab$.

Symmetry: if $((a, b), (c, d)) \in R$ then $ad = bc$ so $cb = da$ so $((c, d), (a, b)) \in R$.

Transitivity: if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ then $ad = bc$ and $cf = de$. Since all integers are positive and hence non-zero, we have $d = cf/\epsilon$ so $acf/\epsilon = be$ so $afc = be\epsilon$ and thus $af = be$. So $((a, b), (e, f)) \in R$.

9.5.22 Yes. It is reflexive (all vertices have arrows from and to themselves), symmetric (all arrows have a corresponding arrow in the opposite direction) and transitive (check by hand that any time you follow two arrows, there is a single arrow with the same start and end).
9.5.46

(a) yes
(b) yes
(c) no (the sets overlap at the integers)
(d) no (the integers are not a part of the union of the sets)
(e) yes
(f) yes

9.5.49 See back of the book

9.5.68 Call the $n$ element set $S$ and consider a particular element $x$ of $S$. In any partition $S = \bigcup S_i$, $x$ belongs to exactly one subset $S_i$; without loss of generality we can number the subsets of any partition such that $x$ always belongs to $S_1$. Now the number of partitions in which $|S_1| = j+1$ is the number of ways to choose $j$ other elements of $S$ to belong to $S_1$ along with $x$, times the number of ways to partition the remaining $n-j-1$ elements, i.e. \( \binom{n-1}{j} p(n-j-1) \). Since $S_1$ could be any size from 1 to $n$, the total number of partitions of $S$ is the sum over all possible sizes of $S_1$ of the number of partitions of $S$ with $S_1$ having that size. This is exactly \( \sum_{j=0}^{n-1} \binom{n-1}{j} p(n-j-1) \) as desired.