Chapter 7.4

10 The space of possible outcomes is (1=heads, 0 = tails)

\[ S = \{00\} \cup \{100, 010\} \cup \{1100, 1010, 0110\} \cup \{11100, 11010, 10110, 01110\} \cup \{111111, 111110, 111101, 111100, 111011, 110110, 110111, 110110, 101111, 101110, 011111, 011110, 011110\} \]

where each state of length \( k \) has probability \( 2^{-k} \). Let \( X : S \to \mathbb{R} \) be the random variable “number of flips.” Then

\[ \mathbb{E} X = 1 \times 2 \times \frac{1}{2^2} + 2 \times 3 \times \frac{1}{2^3} + 3 \times 4 \times \frac{1}{2^4} + 4 \times 5 \times \frac{1}{2^5} + 12 \times 6 \times \frac{1}{2^6} \]

12 The probability of not rolling 6 \( k \) times and then rolling 6 is \( \left(\frac{5}{6}\right)^k \times \frac{1}{6} \). This means the random variable \( X = \) “number of times before rolling six” is geometrically distributed with parameter \( \frac{1}{6} \). Theorem 4 says then that \( \mathbb{E} X = 6 \).

16 Two random variables \( X \) and \( Y \) are by definition independent if the events \( \{X = r_1\} \) and \( \{Y = r_2\} \) are independent for all \( r_1, r_2 \). So we need only find one example of \( r_1 \) and \( r_2 \) for which these are not independent events. Consider \( r_1 = r_2 = 0 \). The events \( \{X = 0\} \) and \( \{Y = 0\} \) each have probability \( \frac{1}{4} \) of occuring, while the event \( \{X = 0\} \cap \{Y = 0\} \) (which says we flipped 0 heads and 0 tails in 2 flips) has probability 0.

24 This is an exercise in unwinding the definitions: If \( S \) is the space of possible outcomes,

\[ \mathbb{E}(I_A) := \sum_{s \in S} P(s) \times I_A(s) = \sum_{s \in A} P(s) \times I_A(s) + \sum_{s \notin A} P(s) \times I_A(s) \]

\[ \sum_{s \in A} P(s) \times 1 + \sum_{s \notin A} P(s) \times 0 = \sum_{s \in A} P(s) =: P(A) \]

25 Solution in back of book.

32 Consider a random variable \( X \) such that \( \text{Var}(X) > 0 \), and let \( Y = -X \). These are clearly not independent random variables. We compute

\[ \text{Var}(X + (-X)) = \text{Var}(0) = 0. \]

(Note that the 0 in \( \text{Var}(0) \) is still a random variable, it is the constant zero function from \( S \) to \( \mathbb{R} \).) On the other hand,

\[ \text{Var}(X) + \text{Var}(-X) = 2\text{Var}(X) > 0. \]

36 Our random variable is \( X = \) “number of tails.” We need to know the variance of \( X \) to use Chebyshev’s inequality. The variance of a single coin flip is \( \frac{1}{4} \) (e.g. by example 14). As \( X \) is the sum of \( n \) such independent random variables (the indicator function for each individual coin flip), Benaymè’s formula says that \( \text{Var}(X) = n/4 \). Now Chebyshev’s formula says that

\[ P \left( |X(s) - \mathbb{E}X| \geq \sqrt{n} \right) \leq \text{Var}(X)/((\sqrt{n})^2) = \frac{1}{4}. \]

37 Solution in back of book.

38 Consider the random variable \( X = \) “number of bottlesfilled today.”

(a) Assuming the employees don’t drink more bottles than they fill, \( X(s) \geq 0 \). Then Markov’s inequality says that

\[ P(X(s) \geq 11,000) \leq \frac{10,000}{11,000} = \frac{10}{11}. \]
(b) On the other hand, Chebyshev’s inequality says that

\[ P(9,000 \leq X(s) \leq 11,000) = 1 - P(|X(s) - \mathbb{E}X| \geq 1,000) \geq 1 - 1,000/(1,000)^2 = \frac{999}{1,000}. \]

48 Name the random variable \( X = \text{“number of balls which fall in bin 1”} \) as well as the random variables

\[ X_i = \begin{cases} 
1 & \text{ball } i \text{ falls into bin 1} \\
0 & \text{otherwise}
\end{cases} \]

Then

\[ X = \sum_{i=1}^{m} X_i. \]

By linearity of expectations,

\[ \mathbb{E}X = \sum_{i=1}^{m} \mathbb{E}X_i = \frac{m}{n}. \]