Math 55 Spring 2016 Practice Final Exam

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180 minutes, closed book, one page of notes allowed

1. (2 points each) True or False:

   (a) If $P(x, y, z, w)$ is a propositional function, then the propositions

   $$\exists x \forall y \exists z \forall w P(x, y, z, w)$$

   and

   $$\exists x \exists z \forall y \forall w P(x, y, z, w)$$

   are logically equivalent.

   (b) The sum of two positive irrational numbers must be irrational.

   (c) If p is prime, $a \equiv b \pmod{p}$, and $c \equiv d \pmod{p}$ then $a^c \equiv b^d \pmod{p}$.

   (d) The set of all finite simple graphs is uncountable.

   (e) If $X : S \to \mathbb{R}$ is a random variable then the events $E = \{s : X(s) > 0\}$ and $F = \{s : X(s) < 0\}$ are independent.

2. (8 points) Prove or disprove: there exists a prime $p > 3$ such that $p + 2$ and $p + 4$ are also prime.

3. (8 points) Find the set of all solutions x to the system of congruences

   $$x \equiv 2 \pmod{5} \qquad x \equiv 3 \pmod{9}.$$

4. (8 points) Prove or disprove: for all primes p and q such that $p \neq q$, the relation

   $$R = \{(x, y) : \text{x} \equiv \text{y} \pmod{\text{p}} \vee \text{x} \equiv \text{y} \pmod{\text{q}}\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

   is an equivalence relation.

5. (8 points) Let $S = \{1, 2, 3\}$. How many bijections

   $$f : S \to S$$

   are there such that $f \circ f$ is equal to the identity? (i.e., $(f \circ f)(x) = x$ for all $x \in S$) (hint: prove that for every such bijection, there is an element $x \in S$ such that $f(x) = x$.)

6. (8 points) Consider the recursive definition:

$$a_0 = x \quad a_1 = y \quad a_n = a_{n-1}^2 + a_{n-2}.$$

Prove that if x and y are rational then a_n must be rational for all $n \geq 0$.

7. (8 points) Suppose I randomly throw 10 balls into 4 bins (where each ball is thrown independently and uniformly into one of the bins). (a) What is the expected number of empty bins? (b) What is the probability that every bin receives at least one ball?

8. (8 points) Suppose I roll a single die repeatedly until three different faces have come up. What is the expected number of times I need to roll the die? (hint: first figure out the expected number of rolls before two different faces have come up).

9. (9 points) Let a_n be the number of nonempty words made out of the letters X, Y, Z (repetitions allowed) with n letters, which do not contain two consecutive X's. Calculate a_0, a_1, and a_2. Write a linear recurrence relation for a_n. Use it to derive a closed-form expression for the generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

of the sequence a_n.

10. (8 points) Let $n > 1$ be an integer and $A = \{1, \ldots, n\}$ and consider the graph with vertices

$$V = \mathcal{P}(A)$$

and edges

$$E = \{\{S, T\} : |(S \setminus T) \cup (T \setminus S)| = 1\}$$

(a) What are the degrees of the vertices in G? (b) For what values of n is G bipartite? (c) For what values of n does G have an Euler circuit? (d) For what values of n is G connected? Justify your responses with proofs.

11. (8 points) Prove that if a simple graph has degree sequence $2, 2, 2, 1, 1, 1, 1$ then it must be disconnected.

12. (9 points) Suppose $G = (V, E)$ is a connected simple graph and $C = e_1, \ldots, c_m$ is a simple circuit in G. Prove that the graph

$$G - e_1 = (V, E \setminus \{e_1\})$$

obtained by deleting e_1 must also be connected.