Math 55 Final Exam, Prof. Srivastava
Friday, May 13, 2016, 7:10–10:00pm, 100 Lewis Hall.

Name: ______________________________

SID: ______________________________

Instructions: Write all answers in the provided space. Please write carefully and clearly, in complete English sentences. You may leave answers "unsimplified", as sums/products/ratios possibly involving binomial coefficients. This exam includes three pages of scratch paper, which must be submitted, but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page.

You are allowed to bring one letter-size single sided page of notes. You are not allowed to use any other materials or electronic devices.

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All question are ten points, except for 4,5,6 which are worth 8,6,6 points, respectively.

Do not turn over this page until your instructor tells you to do so.

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1. Circle true (**T**) or false (**F**) for each of the following. There is no need to provide an explanation.

   (a) (2 points) The compound proposition

   $$(p \to (q \wedge \neg q)) \to \neg p$$

   is a tautology.

   Solution: True. $q \wedge \neg q \equiv F$, so $p \to F$ can only be true if p is false.

   T **F**

   (b) (2 points) There exist integers x and y such that $21x + 54y = 1$.

   Solution: False. Note that 3 is a common divisor of $21 = 7 \cdot 3$ and $54 = 18 \cdot 3$. If such a linear combination existed, we would have $3|21x + 54y$ or $3|1$, which is absurd.

   One can also simply use Bezout's theorem to say that such a linear combination implies $gcd(21, 54) = 1$, which is false.

   T **F**

   (c) (2 points) A graph is $3-$colorable if and only if all of the circuits in it have length divisible by 3.

   Solution: False. Consider the graph with two vertices and one edge. All of its circuits have even length, but it is 3-colorable.

   T **F**

   (d) (2 points) If E and F are events in a probability space such that $p(E) = 0.99$ and $p(F) = 0.99$ then E and F cannot be independent.

   Solution: False. Consider two independent biased coin flips where each coin has a probability 0.99 of being heads, and let E be the event that the first coin is heads, F that the second coin is heads.

   T **F**

   (e) (2 points) Let I_n denote the number of injective functions from $\{1, 2, 3, \ldots, n\}$ to $\{1, 2, \ldots, 55\}$. If $m \geq n$ then it must be the case that $I_m \geq I_n$.

   Solution: False. Note that $I_m = 0$ for $m > 55$ by the pigeonhole principle, whereas $I_m \neq 0$ for $m \leq 55$.

   T **F**

2. (10 points) Suppose A is a (possibly infinite) set. Prove that A and its power set $\mathcal{P}(A)$ cannot have the same cardinality. (hint: Prove that there is no bijection $f : A \to \mathcal{P}(A)$ by considering for any function f the subset $B = \{x \in A : x \notin f(x)\}$.)

Solution: Assume for the sake of contradiction that A and $\mathcal{P}(A)$ have the same cardinality. By definition, there is a bijection $f : A \to \mathcal{P}(A)$. Consider the set

$$B = \{x \in A : x \notin f(X)\},$$

and note that it is an element of $\mathcal{P}(A)$ since it is by construction a subset of A. Since f is surjective, there must exist an element $y \in A$ such that $f(y) = B$. There are now two possibilities: if $y \in B$, then since we $B = f(y)$ we have $y \in f(y)$, but now $y \notin B$ by the definition of B, which is absurd. On the other hand if $y \notin B$ then $y \notin f(y)$ so $y \in B$, which is also absurd. Thus, our assumption is false, and no such bijection exists, and A and $\mathcal{P}(A)$ must not have the same cardinality.

Some people asked me during the exam if we needed to assume $A \neq \emptyset$. In fact, the argument above works even when A is empty, but if this bothers you, it is also easy to simply handle the case $A = \emptyset$ by observing $\mathcal{P}(A) = \{\emptyset\}$, which has cardinality one, and then assume $A \neq \emptyset$.

This problem was essentially assigned on HW2 and HW3 (2.1.46, 2.5.40).

3. Consider the following one player game. The game begins with n shapes, each of which is either a circle or a square. In each move, I have two options: (A) choose two shapes of the same type and replace them by a square, OR (B) choose two shapes of different types and replace them by a circle. The game is over when there is exactly one shape left, and I win if that shape is a circle (otherwise I lose).

   (a) (4 points) Prove by induction that the game always terminates in a finite number of moves (regardless of the initial state).

   Solution: Let $P(n)$ denote the statement that the $n-$shape game always terminates in $n-1$ steps. We will show that $P(n)$ is true for all integers $n \geq 1$.

   Basis Step: When $n = 1$ the game is already over, so indeed it terminates in zero steps.

   Inductive Step: Assume $P(k)$ is true, and consider the game with $k+1$ shapes. As each move replaces two shapes with one shape, the total number of shapes decreases by one, so after one move the number of shapes remaining is k. The rest of the game is the same as playing with k shapes initially, so by our inductive hypothesis it terminates in $k-1$ steps. Adding the first step, we conclude that the $k+1$-shape game terminates in $1+k-1 = k = (k+1)-1$ steps, as desired.

   (b) (6 points) Prove that I win the game if and only if the initial number of circles is odd.

   Solution: ($\Leftarrow$). Assume that the initial number of circles is odd. Observe that a move of type A reduces the number of circles by either zero or two (depending on whether we remove two squares or two circles), and a move of type B does not change the nmber of circles. Thus, every move preserves the property that the total number of circles is odd. Since the game terminates after $n-1$ moves by part (1), the number of circles in the terminating configuration must also be odd, so in particular the last shape must be a circle, which means that I win the game.

   ($\Rightarrow$). Assume that the initial number of circles is even. By the same argument as above, each move preserves the parity (even or odd) of the number of circles. Thus, after $n-1$ moves the number of circles is still even; since there is only one shape left after this many moves, it must be a square, so I lose the game.

   The above arguments can also be formalized by induction, which is essentially implicit in the wording "every move preserves the parity" and "the number of moves is finite". If you want to do this, an appropriate predicate to use would be something like: $P(n)$ = I win the game with n shapes iff the number of circles is odd.

4. (8 points) For any prime p, consider the following relation on $S = \{1, \dots, p-1\}$:

$$R_p = \{(a,b) : a = b \quad \vee ab \equiv 1 \pmod{p}\} \subseteq S \times S.$$

Prove or disprove: For every prime p, R_p is an equivalence relation.

Solution: This statement is true. Assume p is a prime. We will show that R_p is reflexive, symmetric, and transitive:

Reflexive: Observe that for every $a \in S$ we have $a = a$ so $(a,a) \in R_p$.

Symmetric: Assume $(a,b) \in R_p$. If $a = b$ then $b = a$ so $(b,a) \in R_p$. Otherwise, $ab \equiv 1 \pmod{p}$, and since multiplication modulo p is commutative we have $ba \equiv 1 \pmod{p}$, so again $(b,a) \in R_p$, as desired.

Transitive: Assume $(a,b) \in R_p$ and $(b,c) \in R_p$. We will show that $(a,c) \in R_p$. If $a = b$ or $b = c$ then this is immediate, so assume $a \neq b$ and $b \neq c$. By the definition of R_p we have

$$ab \equiv 1 \pmod{p} \qquad \text{and} \qquad bc \equiv 1 \pmod{p}.$$

Since p is prime every number in S must have an inverse modulo p (since in particular each such number is relatively prime to p), in particular b must have an inverse, call it z. Multiplying both sides of the above congruences by z, we have

$$a(bz) \equiv z \pmod{p} \Rightarrow a \equiv z \pmod{p},$$

$$(bz)c \equiv z \pmod{p} \Rightarrow c \equiv z \pmod{p}.$$

Thus, we conclude that $a \equiv c \pmod{p}$, which means $p | a - c$. However since $a, c \in S$ we have $|a - c| < p$, so we must have $a = c$, which implies $(a,c) \in R_p$, as desired.

5. (6 points) Calculate the remainder $(-55)^{2016}$ **mod** 13.

Solution: Observe that

$$-55 = -13 \cdot 4 - 3 \equiv -3 \equiv 10 \pmod{13}.$$

Since 13 is prime we can try to use Fermat's Little Theorem, which says $a^{12} \equiv 1 \pmod{13}$ whenever $13 \not| a$, and we have $13 \not| 10$. By long division we observe that $2016 = 12 \cdot 168$,, so the expression of interest becomes

$$(10)^{12 \cdot 168} \equiv (10^12)^{168} \equiv 1^{168} \equiv 1 \pmod{13}.$$

Thus, the remainder is 1.

6. (6 points) Suppose $k \geq 1$ and $(x_1, \ldots, x_k)$ is a randomly chosen $k-$permutation of $\{1, \ldots, n\}$ (i.e., an ordered arrangement of k distinct elements, chosen uniformly from all such arrangements). What is the probability that it is a strictly increasing sequence, i.e., that

$$x_1 < x_2 < \ldots < x_k?$$

Solution: Observe that a random $k-$permutation is generating by choosing a random $k-$combination, and then choosing a random permutation of the k chosen numbers. Since all permutations are equally likely and only one of them is increasing, the probability is $1/k!$.

If you want to do it a bit more mechanically, with sample spaces etc: the sample space S is the set of all of $k-$permutations of $\{1, \ldots, n\}$, so we have $|S| = P(n, k) = \frac{n!}{k!}$. The event E is the set of increasing $k-$permutations. Observe that the set of increasing $k-$permutations is in bijection with the set of k-combinations: to choose such a permutation we simply choose k (unordered) elements of $\{1, \ldots, n\}$ and then sort them in increasing order; moreover, every increasing permutation uniquely determines a $k-$combination. Thus, $|E| = \binom{n}{k}$, and the desired probability is

$$\frac{P(n,k)}{\binom{n}{k}} = \frac{1}{k!}.$$

7. (10 points) Consider a dial having a pointer that is equally likely to point to each of n regions numbered $1, 2, \ldots, n$. If I spin the dial 3 times, what is the probability that the sum of the selected numbers is exactly n?

Solution: The sample space of possible outcomes is $S = \{(x_1, x_2, x_3) : x_i \in \{1, \ldots, n\}\}$, where x_i is the number on the ith spin, and we have

$$|S| = n \cdot n \cdot n = n^3.$$

The event of interest is the set

$$E = \{(x_1, x_2, x_3) \in S : x_1 + x_2 + x_3 = n\}.$$

To find the cardinality of E, we observe that it is the set of integer solutions to the equation $x_1 + x_2 + x_3 = n$ with the constraints $1 \leq x_i \leq n$. This is the same as putting $n-3$ indistinguishable balls into 3 distinguishable bins (since each bin must have at least one, we put these in in the beginning), so by stars and bars the number of ways to do it is

$$\binom{n-3+3-1}{2} = \binom{n-1}{2}.$$

Thus, the required probability is

$$\frac{|E|}{|F|} = \frac{\binom{n-1}{2}}{n^3}.$$

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8. Suppose I roll a standard 6−faced die 7 times. Let X be a random variable equal to the number of **distinct** faces that appear (a face is a number in $\{1, 2, 3, 4, 5, 6\}$.)

   (a) (6 points) What is the expectation of X?

Solution: Let X_i be the indicator random variable which is 1 iff the face i is seen, and notice that

$$X = X_1 + \ldots + X_6.$$

By linearity of expectation, we have

$$\mathbb{E}X = \mathbb{E}X_1 + \ldots + \mathbb{E}X_6.$$

For any $i = 1, \ldots, 6$, we have

$$\mathbb{E}X_i = p(\text{face } i \text{ is seen}) = 1 - p(\text{face } i \text{ is not seen}).$$

Since the rolls are independent and the probability of not seeing i on any roll is $5/6$, the latter probability is

$$1 - (5/6)^7.$$

Thus, by linearity, the desired expectation is

$$6(1 - (5/6)^7).$$

The key here is to use linearity of expectation — it is quite a bit harder if you try to do it directly. This problem is essentially identical to problem 7a on the practice final.

   (b) (4 points) What is the variance of X?

Solution: *This was probably the hardest question on the exam and involved combining linearity of expectation, taking complements, the definition of variance, counting, and inclusion-exclusion.* Recall that

$$V(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

We have already calcuated

$$(\mathbb{E}X)^2 = 36(1 - (5/6)^7)^2$$

in the previous part, so it remains to calculate $\mathbb{E}X^2$. Observe that

$$\mathbb{E}(X_1 + \ldots + X_6)^2 = \sum_{i=1}^{6} \mathbb{E}X_i^2 + \sum_{1 \le i \ne j \le 6} \mathbb{E}X_i X_j.$$

Since $X_i^2 = X_i$ because X_i is an indicator we have

$$\sum_{i=1}^{6} \mathbb{E}X_i^2 = \sum_{i=1}^{6} \mathbb{E}X_i = \mathbb{E}X = 6(1 - (5/6)^7),$$

from the previous calculation.

We now calculate the remaining terms. Let E_i denote the event that the face i is seen in the 7 rolls. Let $i \neq j$ and observe that

$$\mathbb{E}X_iX_j = p(E_i \cap E_j) = 1 - p(\bar{E}_i \cup \bar{E}_j) == 1 - (p(\bar{E}_i) + p(\bar{E}_j) - p(\bar{E}_i \cap \bar{E}_j),$$

by inclusion-exclusion. *Remark: The reason we are doing this is that E_i and E_j are not independent, and it is easier to calculate $p(\bar{E}_i \cap \bar{E}_j)$ than $p(E_i \cap E_j)$.* We calculate

$$p(\bar{E}_i \cap \bar{E}_j) = (4/6)^7,$$

so

$$\mathbb{E}X_iX_j = 1 - (2(5/6)^7 - (4/6)^7).$$

Since this calculation holds for any $i \neq j$ and there are $2 \cdot \binom{6}{2}$ such pairs, we obtain

$$\mathbb{E}X^2 = 6(1 - (5/6)^7) + 2 \cdot \binom{6}{2} \left(1 + (4/6)^7 - 2(5/6)^7\right),$$

so in all the variance is

$$6(1 - (5/6)^7) + 2 \cdot \binom{6}{2} \left(1 + (4/6)^7 - 2(5/6)^7\right) - 36(1 - (5/6)^7)^2.$$

9. (10 points) Consider the sequence $a_0, a_1, a_2, \ldots$ defined by the formula $a_n = 3^n + 4^{n+2}$. Derive a closed form expression for the generating function

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

of this sequence, i.e., find polynomials $p(x)$ and $q(x)$ such that $G(x) = p(x)/q(x)$.

Solution: We can write each term as $a_n = b_n + c_n$ for $b_n = 3^n$ and $c_n = 4^{n+2}$. The generating function for b_n is

$$F(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + 3x + 3^2x^2 + \ldots + 3^n x^n + \ldots = \frac{1}{1-3x},$$

and the generating function for c_n is

$$H(x) = \sum_{n=0}^{\infty} c_n x^n = 4^2(1 + 4x + 4^2x^2 + \ldots) = \frac{16}{1-4x}.$$

Thus, adding term by term, we can write

$$G(x) = F(x) + H(x) = \frac{1}{1-3x} + \frac{16}{1-4x} = \frac{1 - 4x + 16 - 48x}{(1-3x)(1-4x)},$$

for polynomials $p(x) = 17 - 52x$ and $q(x) = (1-3x)(1-4x)$.

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10. Prove or disprove:

(a) (5 points) There exists a simple graph with 6 vertices and 5 edges such that G has an Euler circuit.

Solution: The question should have said: "there exists a **connected** simple graph with 6 vertices ind 5 edges", which is much more interesting than just asking for a simple graph. It was my mistake to forget this important restriction, and the question becomes really easy if G is not required to be connected. Those who noticed this and were confident that they knew the definition of an Euler circuit (a circuit which visits every **edge** of the graph exactly once, but not necessarily every vertex) earned an easy 5 points by giving the example of a disconnected graph, such as the union of a $5-$cycle and a single isolated vertex. This graph indeed has 6 vertices and 5 edges, and a circuit which traverses each edge exactly once.

We will also give full points to everyone who assumed that G has to be connected in which case no such graph exists. Here is the proof: Assume G is a simple connected graph with $|V| = 6$ and $|E| = 5$, and assume for contradiction that G has an Euler circuit. By the theorem proved in class, this means that G must be connected and all vertices must have even degree; in particular, the degree of each vertex must be at last two, since a degree of vertex zero would cause G to be disconnected. Thus, the sum of the degrees of all the vertices must be at least 12. However, by the handshaking theorem, we must have $\sum_{v\in V}\deg(v) = 2|E| = 10$, a contradiction.

(b) (5 points) There exists a simple graph G with 6 vertices and 7 edges such that G has an Euler circuit.

Solution: True. Consider the graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{12, 23, 31, 24, 45, 56, 62\}$. *One might arrive at such a graph by trying to repeat the argument of the previous part, and realize that such a graph must have degree sequence $2, 2, 2, 2, 2, 4$, which leads to the graph above.*

11. (10 points) Prove by induction on k that every simple graph with n vertices and k edges has at least $n - k$ connected components.

Solution: We proceed by induction. Fix $n \geq 1$ and let $P(k)$ be the statement that every simple graph with n vertices and k edges has at least $n - k$ connected components. We will show that $P(k)$ is true for all $k \geq 0$.

Basis Step: $P(0)$ says that a graph with no edges has n connected components, and this is true since each vertex is its own connected component.

Inductive Step: Assume $P(k)$ is true. Let G be a graph with n vertices and $k+1$ edges. Let $e = uv$ be an arbitrary edge of G, and let $G' = G - e$ be the graph obtained by deleting e. Let $G_1, \ldots, G_\ell$ be the connected components of G' and note that $\ell \geq n - k$ by the inductive hypothesis. Notice that if u and v are in the same connected component of G', then adding it does not change the connected components, so in this case G also has ℓ connected components. On the other hand if $u \in G_i$ and $v \in G_j$ for some $i \neq j$ then $G_i \cup G_j$ is a connected component of G, since there is a path from every $x \in G_i$ and $y \in G_j$ by concatenating a path from x to u, the edge uv, and a path from v to y. The other connected components of G' remain connected components in G. Thus, the number of connected components decreases by at most one upon adding e, so the number of connected components in G is at least $\ell - 1 = n - k - 1 = n - (k+1)$, as desired.