Correction to Practice Final Number 9

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The following problem is number 9 on the practice exam:

> Let a_n be the number of nonempty words made out of the letters X, Y, Z (repetitions allowed) with n letters, which do not contain two consecutive X's. Calculate a_0, a_1, and a_2. Write a linear recurrence relation for a_n. Use it to derive a closed-form expression for the generating function
>
> $$G(x) = \sum_{n=0}^{\infty} a_n x^n$$
>
> of the sequence a_n.

My solution to this problem in the video and in the accompanying slides contains an error at 1:36 / slide 34-35. The issue is that the recurrence

$$a_n = 2a_{n-1} + 2a_{n-2}$$

only applies when $n - 2 \geq 1$, i.e., when $n \geq 3$, because it is based on a bijection between the set

$$C = \{\text{nonempty words of length } n \text{ with no XX ending in YX}\}$$

and the set of *nonempty* words with no XX of length $n-2$ (and a similar bijection for the set D of words ending in ZX). The problem is that when $n = 2$ the latter set is empty because there are no nonempty words of length zero, whereas the sets C and D contain exactly one element each (namely $C = \{YX\}, D = \{ZX\}$).

Thus, the recurrence only applies when $n \geq 3$ (as opposed to $n \geq 2$, which is what I said in the video). So the right thing to do when transforming the recurrence into a functional equation satisfied by the generating function is to subtract off the first *three* terms $a_0 + a_1x + a_2x^2$ from $G(x)$ (rather than just the first two terms), the first *two* terms from $2xG(x)$, and the *first* term from $2x^2G(x)$. These subtracted terms correspond to the basis step of the recursive definition, and the remaining terms are related by the recursive step.

With this correction made, the second half of slide 36 should say:

$$\begin{aligned}
G(x) - a_0 - a_1x - a_2x^2 &= a_3x^3 + a_4x^4 + \ldots + a_nx^n + \ldots \\
2xG(x) - 2a_0x - 2a_1x^2 &= 2a_2x^3 + 2a_3x^4 + \ldots + 2a_{n-1}x^n + \ldots \\
2x^2G(x)2a_0x^2 &= 2a_1x^3 + 2a_2x^4 + \ldots + 2a_{n-2}x^n + \ldots
\end{aligned}$$

We now note that the first series is the sum of the other two (term by term), which gives the equation:

$$(G(x) - a_0 - a_1x - a_2x^2) = (2xG(x) - 2a_0x - 2a_1x^2) + (2x^2G(x) - 2a_0x^2).$$

Plugging in the values $a_0 = 0, a_1 = 3, a_2 = 8$, we have

$$G(x) - 3x - 8x^2 = (2xG(x) - 6x^2) + 2x^2G(x),$$

which upon rearranging gives

$$(1 - 2x - 2x^2)G(x) = 3x + 2x^2$$

so we have the closed form

$$G(x) = \frac{3x + 2x^2}{1 - 2x - 2x^2},$$

which is the correct answer, as opposed to what is written in the slides.

My apologies for the mistake, and many thanks to Sean Zhou for pointing it out.