MATH 54 FIRST MIDTERM EXAM, PROF. SRIVASTAVA
SEPTEMBER 24, 2018, 5:10PM–6:30PM, 150 WHEELER.

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SID: ______________________

INSTRUCTIONS: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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| Question | Points |
|---|---|
| 1 | 20 |
| 2 | 20 |
| 3 | 10 |
| 4 | 6 |
| 5 | 12 |
| 6 | 10 |
| 7 | 22 |
| Total: | 100 |

Do not turn over this page until your instructor tells you to do so.

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If two nonzero vectors $v_1$ and $v_2$ are linearly dependent then there is a scalar $c$ such that $v_1 = cv_2$.

   **Solution:** True. If $c_1v_1 + c_2v_2 = 0$ and both vectors are nonzero then $c_1$ and $c_2$ must also be nonzero, so $v_1 = (c_2/c_1)v_2$.

   **T** **F**

   (b) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 1 \\ x_2 + 2 \end{bmatrix}$$

   is linear.

   **Solution:** False. $T(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 0$, so $T$ cannot be linear.

   **T** **F**

   (c) If the vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ are linearly independent and the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^5$ is one to one, then $T(v_1), T(v_2), T(v_3)$ must also be linearly independent.

   **Solution:** True. If $c_1T(v_1) + \ldots + c_3T(v_3) = 0$ then by linearity $T(c_1v_1 + \ldots + c_3v_3) = 0$, which since $T$ is one to one implies $c_1v_1 + c_2v_2 + c_3v_3 = 0$. But now since $v_1, v_2, v_3$ are linearly independent we must have $c_1 = c_2 = c_3 = 0$, which means $T(v_1), \ldots, T(v_3)$ are linearly independent.

   **T** **F**

   (d) If the vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ are linearly independent and the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto, then $T(v_1), T(v_2), T(v_3)$ must also be linearly independent.

   **Solution:** False. Consider the vectors $e_1, e_2, e_3 \in \mathbb{R}^4$ and the transformation $T$ defined by $T(e_1) = e_1$, $T(e_2) = e_1$, $T(e_3) = e_2$ and $T(e_4) = e_3$ (i.e., the standard matrix has columns $e_1, e_2, e_2, e_3$) which is onto but since $T(e_1) = T(e_2)$ their images are linearly dependent.

   **T** **F**

(e) If $\text{span}\{v_1, \ldots, v_k\} = \mathbb{R}^n$ and the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, then $\text{span}\{T(v_1), \ldots, T(v_k)\} = \mathbb{R}^m$.

> **Solution:** True. For every $b \in \mathbb{R}^m$ there is an $x \in \mathbb{R}^n$ such that $T(x) = b$. Writing $b = c_1 v_1 + \ldots + c_k v_k$ for some coefficients $c$, we therefore have $T(c_1 v_1 + \ldots + c_k v_k) = b$, which by linearity means $c_1 T(v_1) + \ldots + c_k T(v_k) = b$. Thus we have expressed an arbitrary $b \in \mathbb{R}^m$ as a linear combination of $T(v_1), \ldots, T(v_k)$, so they must span $\mathbb{R}^m$.

**T F**

(f) If the linear systems $Ax = b_1$ and $Ax = b_2$ are consistent then the system $Ax = b_1 + 2b_2$ must be consistent.

> **Solution:** True. Suppose $x_1$ and $x_2$ are vectors such that $Ax_1 = b_1$ and $Ax_2 = b_2$. Then $A(x_1 + 2x_2) = Ax_1 + 2Ax_2 = b_1 + 2b_2$, so $Ax = b_1 + 2b_2$ is also consistent.

**T F**

(g) If the reduced row echelon forms of two $n \times n$ matrices $A$ and $B$ are equal to the identity, then the RREF of the matrix $AB$ is also equal to the identity.

> **Solution:** True. The condition implies that $A$ and $B$ are invertible. Since the product of two invertible matrices is invertible (which can be seen conceptually by composing the inverse linear transformations, or algebraically by multiplying $AB$ by $B^{-1}A^{-1}$), $AB$ must be invertible, which implies that its RREF is equal to the identity.

**T F**

(h) If $A$ and $B$ have the property that $\text{Col}(B) \subseteq \text{Nul}(A)$ then $AB = 0$.

> **Solution:** True. For every vector $x$ we have $Bx \in \text{Col}(B) \subset \text{Nul}(A)$, so $ABx = 0$. This means $AB$ must be the zero matrix (can check this concretely by plugging the standard basis vectors for $x$, which shows that the columns of $AB$ are zero).

**T F**

(i) Any two linearly independent vectors in $\mathbb{R}^2$ form a basis of $\mathbb{R}^2$.

> **Solution:** True. If two vectors are linearly independent, then the matrix with them as columns has a pivot in every column, and since it is square, also in every row. Thus the vectors must also span $\mathbb{R}^2$, so they are a basis.

**T F**

(j) If $\mathcal{B} = \{b_1, \ldots, b_k\}$ is a basis of a subspace $H \subseteq \mathbb{R}^m$ then for every $v \in H$ the coordinate vector $[v]_{\mathcal{B}}$ is an element of $\mathbb{R}^k$.

**Solution:** True. $[v]_{\mathcal{B}}$ is the vector whose entries are the unique coefficients in the linear combination $x = c_1 b_1 + \ldots + c_k b_k$. Since there are $k$ of these coefficients, it is a vector in $\mathbb{R}^k$.

**T F**

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (4 points) A linear system with 3 equations in 2 variables which is consistent.

   > **Solution:** There are infinitely many examples, e.g. repeating the same equation three times:
   > $$x_1 + x_2 = 1$$
   > $$x_1 + x_2 = 1$$
   > $$x_1 + x_2 = 1$$
   > or multiples of the same equation, or two consistent equations and another copy of one of them, or the all zero system, or any homogeneous system. In terms of row reduction, such a system will have no pivot in the augmented column.

   (b) (4 points) A linear system with 2 equations in 3 variables which is not consistent.

   > **Solution:** For instance:
   > $$x_1 + x_2 + x_3 = 0$$
   > $$x_1 + x_2 + x_3 = 1.$$
   > In terms of row reduction, such a system will have a pivot in the augmented column.

   (c) (4 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which is both one to one and onto.

   > **Solution:** Such a $T$ does not exist. There are many proofs, but one is that the standard matrix of $A$ is a $3 \times 2$ matrix. Since it has more columns than rows, it cannot have a pivot in every column, so the corresponding system $Ax = b$ cannot have a unique solution for any $b$, which means $T$ is not one to one.

[Scratch Paper 1]

(d) (4 points) A $2 \times 3$ matrix $A$ and a $3 \times 2$ matrix $B$ such that $AB$ is invertible.

**Solution:** Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

for which $AB = I$, which is clearly invertible.

(e) (4 points) A $2 \times 3$ matrix $A$ and a $3 \times 2$ matrix $B$ such that $BA$ is invertible.

**Solution:** No such $A$ and $B$ exist. This is because since $A$ has three columns and two rows, it must have a free variable, so there is a nonzero $x$ such that $Ax = 0$. But now $BAx = 0$, so $BA$ cannot be invertible.

In terms of linear transformations, multiplication by $A$ is not one to one since it maps $\mathbb{R}^3$ to $\mathbb{R}^2$, so there are two distinct $x_1$ and $x_2 \in \mathbb{R}^3$ such that $Ax_1 = Ax_2$. But now $BAx_1 = BAx_2$, so multiplication by $BA$ is not one to one either, and cannot be invertible.

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3. (10 points) For which values of $s \in \mathbb{R}$ are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}.$$

Row reducing:

$$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ 3 & 8 & s \end{bmatrix} \underset{\substack{R_2+R_1 \\ R_3-3R_1}}{\sim} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & 23 & s-3 \end{bmatrix} \underset{R_3 - \frac{23}{2}R_2}{\sim} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & s-26 \end{bmatrix},$$

we see that the REF has a pivot in every col (i.e., no free variables) exactly when $\boxed{s \neq 26}$. So the vectors are linearly indep for $s \neq 26$.

4. (6 points) State precisely the definition of an *onto* transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

**Solution:** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if for every $b \in \mathbb{R}^m$, there is at least one $x \in \mathbb{R}^n$ such that $T(x) = b$.

5. (12 points) Consider the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}.$$

Find the first vector in this set which is in the span of the other ones, and express it as a linear combination of them. Explain your reasoning.

6. (10 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\5\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\8\\0\end{bmatrix}.$$

Find the value of $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)$.

**Solution:** We first find a linear combination expressing $\begin{bmatrix}-1\\2\end{bmatrix}$ in terms of the vectors for which we know the value of $T$,

$$\begin{bmatrix}-1\\2\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix}.$$

The coefficients can be found by solving the augmented system $\begin{bmatrix}1 & 1 & -1\\0 & 1 & -2\end{bmatrix}$, which gives $c_1 = -3, c_2 = 2$. By linearity we now have:

$$T(\begin{bmatrix}-1\\2\end{bmatrix}) = T(-3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}) = -3T(\begin{bmatrix}1\\0\end{bmatrix}) + 2T(\begin{bmatrix}1\\1\end{bmatrix}) = -3\begin{bmatrix}0\\5\\4\end{bmatrix} + 2\begin{bmatrix}2\\8\\0\end{bmatrix} = \begin{bmatrix}4\\1\\-12\end{bmatrix}.$$

7. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by:

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix}$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the geometric linear transformation which reflects a point $x \in \mathbb{R}^2$ across the line $x_1 = x_2$.

(a) (6 points) Show that $T_2$ is invertible and find its inverse transformation $T_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

**Solution:** Conceptual proof: reflecting a point *twice* across the line $x_1 = x_2$ gives back the same point, i.e., $T_2 \circ T_2(x) = x$ for every $x \in \mathbb{R}^2$. Thus $T_2$ is invertible and its inverse is itself, i.e., $T_2^{-1} = T_2$. The matrix of the inverse is therefore the same as the standard matrix of $T_2$, which has columns $T_2(e_1) = e_2$ and $T_2(e_2) = e_1$, so it is

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

More concrete proof: First find the standard matrix, which is the matrix $B$ as above. Row reducing this matrix augmented with the identity, we find that $B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so the standard matrix of $T_2^{-1}$ is also

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) (6 points) Find the standard matrix of the composition $T = T_2^{-1} \circ T_1$. Call this matrix $A$.

**Solution:** The columns of the standard matrix are given by

$$T_2^{-1}(T_1(e_1)) = T_2^{-1}\left(\begin{bmatrix} 1-0 \\ 0-0 \end{bmatrix}\right) = B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$T_2^{-1}(T_1(e_2)) = T_2^{-1}\left(\begin{bmatrix} 0-1 \\ 1-0 \end{bmatrix}\right) = B\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$T_2^{-1}(T_1(e_3)) = T_2^{-1}\left(\begin{bmatrix} 0-0 \\ 0-1 \end{bmatrix}\right) = B\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

so it is

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

(c) (4 points) Find a basis for Col($A$).

**Solution:** The REF of $A$ (after one swap) is

$$R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

The first two columns of $R$ are pivot columns, so the corresponding columns of $A$, namely

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\},$$

form a basis for Col($A$). *Note that in this case* $\mathrm{Col}(A) = \mathbb{R}^2$ *so any basis for* $\mathbb{R}^2$ *is a valid answer. However it is not valid to reason that the first two columns of* $R$ *are a basis of* Col($A$) *— this is not true in general, they just happen to be in this particular case since* $\mathrm{Col}(A) = \mathrm{Col}(R) = \mathbb{R}^2$.

(d) (4 points) Find a basis for Nul($A$).

**Solution:** The RREF of $A$ (after the row operation $R_1 + R_2$ is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix},$$

which expresses the pivot variables as $x_1 = x_3, x_2 = x_3$. Thus,

$$\mathrm{Nul}(A) = \left\{ x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

so this set is a basis for the null space.

(e) (2 points) Is $T$ one to one? Explain in terms of your previous answers.

**Solution:** No it is not — since Nul($A$) contains infinitely many vectors, there is more than one solution to $T(x) = 0$, which means $T$ is not one to one.