Math 54 First Midterm Exam, Prof. Srivastava
September 24, 2018, 5:10pm–6:30pm, 150 Wheeler.

Name: ____

SID: ____

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

UC Berkeley Honor Code: *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

Sign here: ____

Question	Points
1	20
2	20
3	10
4	6
5	12
6	10
7	22
Total:	100

Do not turn over this page until your instructor tells you to do so.

Name and SID: ____________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If two vectors v_1 and v_2 are linearly dependent then there is a scalar c such that $v_1 = cv_2$. **T F**

   (b) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

   $$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 1 \\ x_2 + 2 \end{bmatrix}$$

   is linear.

   T F

   (c) If the vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ are linearly independent and the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^5$ is one to one, then $T(v_1), T(v_2), T(v_3)$ must also be linearly independent.

   T F

   (d) If the vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ are linearly independent and the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto, then $T(v_1), T(v_2), T(v_3)$ must also be linearly independent.

   T F

   (e) If $\text{span}\{v_1, \ldots, v_k\} = \mathbb{R}^n$ and the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, then $\text{span}\{T(v_1), \ldots, T(v_k)\} = \mathbb{R}^m$.

   T F

   (f) If the linear systems $Ax = b_1$ and $Ax = b_2$ are consistent then the system $Ax = b_1 + 2b_2$ must be consistent.

   T F

   (g) If the reduced row echelon forms of two $n \times n$ matrices A and B are equal to the identity, then the RREF of the matrix AB is also equal to the identity.

   T F

   (h) If A and B have the property that $\text{Col}(B) \subseteq \text{Nul}(A)$ then $AB = 0$.

   T F

   (i) Any two linearly independent vectors in $\mathbb{R}^2$ form a basis of $\mathbb{R}^2$.

   T F

   (j) If $\mathcal{B} = \{b_1, \ldots, b_k\}$ is a basis of a subspace $H \subseteq \mathbb{R}^m$ then for every $v \in H$ the coordinate vector $[v]_{\mathcal{B}}$ is an element of $\mathbb{R}^k$. **T F**

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (4 points) A linear system with 3 equations in 2 variables which is consistent.

   (b) (4 points) A linear system with 2 equations in 3 variables which is not consistent.

   (c) (4 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which is both one to one and onto.

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[Scratch Paper 1]

(d) (4 points) A 2×3 matrix A and a 3×2 matrix B such that AB is invertible.

(e) (4 points) A 2×3 matrix A and a 3×2 matrix B such that BA is invertible.

3. (10 points) For which values of $s \in \mathbb{R}$ are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}.$$

4. (6 points) State precisely the definition of an *onto* transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

5. (12 points) Consider the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}.$$

Find the first vector in this set which is in the span of the other ones, and express it as a linear combination of them. Explain your reasoning.

6. (10 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\5\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\8\\0\end{bmatrix}.$$

Find the value of $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)$.

7. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by:

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix}$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the geometric linear transformation which reflects a point $x \in \mathbb{R}^2$ across the line $x_1 = x_2$.

(a) (6 points) Show that T_2 is invertible and find its inverse transformation $T_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

(b) (6 points) Find the standard matrix of the composition $T = T_2^{-1} \circ T_1$. Call this matrix A.

(c) (4 points) Find a basis for Col(A).

(d) (4 points) Find a basis for Nul(A).

(e) (2 points) Is T one to one? Explain in terms of your previous answers.