1. Consider $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0 \right\}$.
   a) If $u$ and $v$ are in $V$, is $u + v$?
      Yes. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $V$. Then we get $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.
      To check that this is in $V$, note that the first coordinate has $u_1 + v_1 \geq 0$ because both $u_1 \geq 0$ and $v_1 \geq 0$ from $u$ and $v$ being in $V$. Similarly, the second coordinate $u_2 + v_2$ is at least zero because both $u_2$ and $v_2$ are at least zero. So $u + v$ satisfies the conditions required to make it in $V$.
   
   b) Show that $V$ is not closed under scaling.
      Any time we scale a vector in $V$ by some negative constant, we will no longer be in $V$.
      For instance, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in $V$ since both coordinates are positive. But we can scale:
      $$-2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$ 
      This is no longer in $V$ since both coordinates are now negative.

2. Consider $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid xy \geq 0 \right\}$.
   a) Is $W$ closed under scaling?
      Yes. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be an element of $W$. Then if we scale $u$ by some real number $c$:
      $$cu = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$ 
      To check that this is still in $W$, we must look at whether $(cu_1)(cu_2)$ is positive. This can be simplified to $c^2(u_1u_2)$. Because $c^2$ is always positive and $u_1u_2 \geq 0$ (due to $u$ being in $W$), we get that $(cu_1)(cu_2) \geq 0$. So $cu$ will still be in $W$.
   
   b) Show that $W$ is not closed under addition.
      We could start with a vector in the first quadrant like $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ that is in $W$. Then we could add to it a vector in the third quadrant, such as $\begin{bmatrix} -1 \\ -10 \end{bmatrix}$, which is also in $W$. Then the sum of these two is $\begin{bmatrix} 1 \\ -8 \end{bmatrix}$, and this is no longer in $W$ because $1(-8) < 0$. 

5. Do polynomials of the form \( p(t) = at^2 \) for any real number \( a \) form a subspace of \( \mathbb{P}_n \)?

**Yes.** We can check that each of our three requirements for being a *subspace* hold for these polynomials.

- **Zero vector:** The zero polynomial is just the polynomial that is zero everywhere, \( p(t) = 0 \). This is of the form \( p(t) = at^2 \) since when \( a = 0 \) we get \( p(t) = 0t^2 = 0 \).

- **Scaling:** If we start with some polynomial \( q(t) = bt^2 \). Then we can scale this polynomial by any real number \( k \). This gives us \( kq(t) = k(bt^2) = (kb)t^2 \). This is still a polynomial of our desired form since it is some real number times \( t^2 \). So starting with a polynomial of the correct form, we end up with a polynomial of the correct form.

- **Addition:** If we start with two polynomials of the correct form, \( p(t) = at^2 \) and \( q(t) = bt^2 \). Then we can add the polynomials \( p(t) + q(t) = at^2 + bt^2 = (a + b)t^2 \). And once again we find that we still have a real number times \( t^2 \). So our set is also closed under adding two polynomials.

6. Do polynomials of the form \( p(t) = a + t^2 \) for any real number \( a \) form a subspace of \( \mathbb{P}_n \)?

**No.** This already fails to be a subspace when we check our first rule. The zero polynomial is not in this set. There is no real number \( a \) we can choose to make \( a + t^2 = 0 \), as there is no way to make the \( t^2 \) term go away.

8. Do all polynomials with \( p(0) = 0 \) form a subspace of \( \mathbb{P}_n \)?

**Yes.** Let’s check the three requirements.

- **Zero vector:** The zero polynomial certainly satisfies the property of \( p(0) = 0 \). It evaluates to 0 everywhere, so in particular it evaluates to 0 at 0.

- **Scaling:** Suppose we start with some polynomial that has \( p(0) = 0 \). Then if we scale it by any real number \( k \), we can see that it still equals 0 at 0. This is because \((k \cdot p)(0) = k \cdot p(0) = k \cdot 0 = 0 \). So our set is closed under scaling.

- **Addition:** Again we want to start with any two polynomials in our set. So suppose \( p(0) = 0 \) and \( q(0) = 0 \). Then the polynomial \((p + q)(t)\) can be evaluated at 0, and we get \( p(0) + q(0) = 0 + 0 = 0 \). So adding two polynomials gives us a polynomial still in our set, i.e. we are closed under addition.

11. Let \( W \) be all of the vectors in \( \mathbb{R}^3 \) of the form \[
\begin{bmatrix}
5b + 2c \\
b \\
c
\end{bmatrix}
\] for any numbers \( b \) and \( c \). Find \( \mathbf{u} \) and \( \mathbf{v} \) that span \( W \). Show that \( W \) is a subspace of \( \mathbb{R}^3 \).

Another way of saying that \( \mathbf{u} \) and \( \mathbf{v} \) will be vectors spanning \( W \) is that anything in \( W \) is some linear combination of \( \mathbf{u} \) and \( \mathbf{v} \). We already know what form the vectors in \( W \) have, and we can manipulate them a bit to get:

\[
\begin{bmatrix}
5b + 2c \\
b \\
c
\end{bmatrix} = b \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.
\]
This then tells us exactly that any vector in $W$ is a linear combination of \[
\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.
\]
So we get that:
\[
W = \text{span} \left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]
This is a subspace of $\mathbb{R}^3$ because the span of vectors is always a subspace.

20. Let $C[a, b]$ be the set of all continuous functions that map from the interval $[a, b]$ to $\mathbb{R}$.

a) What facts must we prove in order to show that this is a subspace of all functions?

- **Zero vector**: The zero vector in this case is the function that is always zero, $f(x) = 0$. (When we add it to any other function, nothing changes.) So we need to prove first that $f(x) = 0$ is continuous.

- **Scaling**: If we start with any continuous function on our interval, $g(x)$. Then we next need to prove that for any real number we have that $kg(x)$ is also continuous.

- **Addition**: Finally, we would prove that if $f(x)$ and $g(x)$ are two continuous functions on $[a, b]$, then $f(x) + g(x)$ is also continuous.

b) Show that $\{f \in C[a, b] \mid f(a) = f(b)\}$ is a subspace of $C[a, b]$.

- **Zero vector**: We can check that the function that is always zero satisfies our property. Since it is zero everywhere, it will be zero at both $a$ and $b$. So the zero function is the same on both end points.

- **Scaling**: Suppose we start with some $f(x)$ that has $f(a) = f(b)$. Then when we scale the function by some number $k$, we get $kf(a) = kf(b)$. So the scaled $kf(x)$ still has the correct property and is in our set.

- **Addition**: Suppose we start with any two functions in our set, $f(x)$ and $g(x)$. Then $f(a) + g(a) = f(b) + g(b)$ will still be true, as $f(a) = f(b)$ and $g(a) = g(b)$. So $f + g$ will also be in our set, i.e. it is closed under addition.

21. Consider $H$ the set of $2 \times 2$ matrices of the form \[
\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}
\]
for any numbers $a, b$ and $d$. Is this a subspace of all $2 \times 2$ matrices?

**Yes.** We can simply check the same three conditions.

- **Zero vector**: In this case, the zero vector for $2 \times 2$ matrices is the matrix with 0 in every entry, \[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Then this is in $H$, as it has the correct form where $a = 0$, $b = 0$, and $d = 0$.

- **Scaling**: If we start with any matrix $M$ in $H$, it will have the form $M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.
Then scaling $M$ by a real number $k$ will give us:
\[
kM = \begin{bmatrix} ka & kb \\ 0 & kd \end{bmatrix}.
\]
This is still in the form of an element of $H$, so $H$ is closed under scaling.
• **Addition:** Start with two matrices $M_1$ and $M_2$ in $H$. When we add them, we get:

$$M_1 + M_2 = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix}.$$ 

Once again, we see that this is of the correct form for an element of $H$, so $H$ is closed under addition.

22. If $F$ is a fixed $3 \times 2$ matrix, let $\mathcal{H}$ be the set of $2 \times 4$ matrices $A$ such that $FA = 0$. Is this a subspace of all $2 \times 4$ matrices?

**Yes.** We can check that our three conditions are satisfied by looking at matrix multiplication and solving a bunch of equations. But instead, let’s consider a more geometric picture and think of these as being linear transformations.

We start with some fixed linear transformation $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Then $\mathcal{H}$ is the set of $A : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ such that $F \circ A$ is the zero linear transformation (it sends anything in $\mathbb{R}^4$ to the zero vector in $\mathbb{R}^3$). Or said another way: $\mathcal{H}$ is the set of linear transformations whose column space is contained in the null space of $F$.

• **Zero vector:** We want to check that the column space of the zero transformation is contained in the null space of $F$. This is true because the column space of the zero transformation is just $\{0\}$, and $0$ will always be in the null space of $F$.

• **Scaling:** If we start with some $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$ whose column space is contained in the null space of $F$, what can we say about the column space of $kA$? We can see that $kA = \begin{bmatrix} ka_1 & ka_2 & ka_3 & ka_4 \end{bmatrix}$. Then any combination of these scaled columns will also be a combination of our original columns. So the column space of $kA$ is contained in the column space of $A$, thus the column space of $kA$ is also contained in the null space of $F$.

• **Addition:** If we start with $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}$ in $\mathcal{H}$.

Then we can look at the column space of:

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 & a_4 + b_4 \end{bmatrix}.$$

Since we can split linear combinations of these columns into parts from $A$ and parts from $B$, anything in the column space of $A + B$ will look like $u + v$ for $u$ in the column space of $A$ and $v$ in the column space of $B$. Then since $A$ and $B$ are in $\mathcal{H}$, we know $F(u) = 0$ and $F(v) = 0$. So we finally get that $F(u + v) = 0$ for anything in the column space of $A + B$, which is exactly what we wanted.

23. Are the following true or false?

a) **False.** A function can equal zero at a single point without equaling zero everywhere. For instance, $f(x) = x - 1$ is zero at $x = 1$, but it is not the zero function.

b) **False.** An arrow in three-dimensional space is an example of a vector, but there are many others that do not look like this. For instance, we can have different dimensions of $\mathbb{R}^n$, or polynomials in $\mathbb{P}_n$.

c) **False.** We need more than just containing the zero vector to be a subspace. We also need the ability to scale and add without leaving the subset.

d) **True.** Checking that something is a vector space in general requires checking a long list (ten items) of properties that must be satisfied. However, if $A$ is a subspace of the
vector space $V$, then $A$ will ‘inherit’ a lot of these properties. For instance, since we can add two vectors in $V$ in any order, this tells us that we can add two vectors in $A$ in any order. In general, from the list at the beginning of section 4.1, properties 2, 3, 7, 8, 9, 10 are all inherited for free by $A$ being a subset of $V$. Checking the remaining properties of $A$ being a vector space are exactly what we check when we show that $A$ is a subspace.

31. If $u$ and $v$ are two vectors in $V$, and $H$ is a subspace of $V$ containing $u$ and $v$, then $H$ also contains span$\{u, v\}$. Also span$\{u, v\}$ is the smallest subspace containing $u$ and $v$.

This is essentially because $H$ is a subspace of $V$, which means $H$ is closed under scaling, addition, and any linear combinations. Since $u$ and $v$ are in $H$, any linear combinations of these two vectors must also be in $H$. But span$\{u, v\}$ is defined to be the set of all linear combinations of $u$ and $v$. So $H$ being closed under linear combinations means it contains everything in span$\{u, v\}$.

This also tells us that span$\{u, v\}$ is the smallest subspace containing $u$ and $v$. We know that span$\{u, v\}$ is already a subspace, since it is closed under any linear combinations. And the above argument shows us that any subspace containing $u$ and $v$ also contains span$\{u, v\}$, i.e. is bigger than the span of these two vectors. So the span of the two vectors must be the smallest such subspace.

32. The intersection of two subspaces is a subspace. But the union of two subspaces might not be a subspace.

If we have $H$ and $K$ two subspaces of $V$, we can check our three requirements to see that $H \cap K$ will also be a subspace of $V$.

- **Zero vector**: Because both $H$ and $K$ are subspaces of $V$, they must both contain 0. And so their intersection $H \cap K$ will also contain the zero vector.
- **Scaling**: If we start with some $v$ that is in $H \cap K$, then $v$ is in both $H$ and $K$. Now consider scaling $v$ to get $kv$. Since both $H$ and $K$ are subspaces closed under scaling, $kv$ will remain in $H$ and it will remain in $K$. So we see that $kv$ is still in both $H$ and $K$, i.e. $v$ is in $H \cap K$.
- **Addition**: Start with two vectors $u$ and $v$ in $H \cap K$. Then $u$ and $v$ will be in both $H$ and $K$. Now because $H$ and $K$ are subspaces closed under addition, we will get that $v + u$ remains in $H$ and remains in $K$. So finally, we get that $v + u$ remains in $H \cap K$.

This shows that the intersection of subspaces will still be a subspace. But we can come up with some simple examples where the union of subspaces is not a subspace. For instance, inside of $\mathbb{R}^2$, we can consider the subspaces of the form $\begin{bmatrix} x \\ x \end{bmatrix}$ and $\begin{bmatrix} x \\ -x \end{bmatrix}$. These are just two lines in the plane. However, the union is not closed under addition. For instance $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ will both be in the union. But adding them gives us $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, which is not in the union.