Chapter 1.8: 1, 4, 8, 12, 14, 16, 17, 22, 24, 31, 32.

1. Since \[
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}
\]
and \[
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}
\]
we have \[T(u) = \begin{pmatrix} 2 \\ -6 \end{pmatrix}\] and \[T(v) = \begin{pmatrix} 2a \\ 2b \end{pmatrix}\].

4. If \(T(x) = \mathbf{b}\) then \(A\mathbf{x} = \mathbf{b}\) and hence the problem reduces to solving the system of linear equations. Reducing the matrix we get the following

\[
\begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]

so \(x_3 = 1\), \(x_2 = 4x_3 - 7 = -3\) and \(x_1 = 6 + 3x_2 - 2x_3 = -5\) and hence the vector we are looking for is

\[
\mathbf{x} = \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}.
\]

8. Five rows and four columns.

12. The vector \(\mathbf{b}\) is in the range of \(A\) if and only if the equation \(A\mathbf{x} = \mathbf{b}\) has a solution. To find out if the system is consistent we do as in exercise 4.

\[
\begin{pmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ -2 & 3 & 0 & 5 & 4 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 3 & -6 & -6 & 4 \\ -2 & 3 & 0 & 5 & 4 \end{pmatrix}
\]

\[
\rightarrow
\begin{pmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ -2 & 3 & 0 & 5 & 4 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & -18 & 11 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}
\]

and hence the system is inconsistent, so \(\mathbf{b}\) is not in the range of the transformation.

14. \(T\) contracts the space by a factor of \(\frac{1}{2}\).
16. $T$ is a reflexion with respect to the axis $x = y$.

17. By the linearity of $T$ we have that $T(3\mathbf{u}) = 3T(\mathbf{u})$, $T(2\mathbf{v}) = 2T(\mathbf{v})$ and $T(3\mathbf{u} + 2\mathbf{v}) = 3T(\mathbf{u}) + 2T(\mathbf{v})$ so

$$T(3\mathbf{u}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad T(2\mathbf{v}) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \quad T(3\mathbf{u} + 2\mathbf{v}) = \begin{bmatrix} 4 \\ 9 \end{bmatrix}. $$

22. (a) True. From the way the product is defined it is clear that if $A$ is a matrix, $\mathbf{u}, \mathbf{v}$ are vectors and $c$ is a scalar, it holds that $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and $A(c\mathbf{u}) = cA\mathbf{u}$.

(b) False. The span of the columns of $A$ is the range, and, since not all linear transformations are surjective, the range does not necessarily coincide with the codomain.

(c) False. There can be more than one solution for the equation $T(\mathbf{x}) = \mathbf{c}$.

(d) True. By definition.

(e) True. By definition.

24. Take an arbitrary $\mathbf{x} \in \mathbb{R}^n$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, there exist weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, and hence

$$T(\mathbf{x}) = T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = 0. $$

31. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set there exist weights $c_1, c_2, c_3$, not all of them 0, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. On the other hand, since $T$ is linear it must satisfy that $T(\mathbf{0}) = \mathbf{0}$. Hence

$$\mathbf{0} = T(\mathbf{0}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3). $$

So, by definition, the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

32. Proceed by contradiction and assume that $T$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$, but, on the other hand

$$\mathbf{0} = T(0, 0) = T[(0, 1) + (0, -1)] \neq T(0, 1) + T(0, -1) = (0, 2), $$

which yields a contradiction.

**Chapter: 1.9: 4,6,9,23abcd,29, 30, 33,36.**

4. By Theorem 10 in chapter 1.9 we see that the matrix is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. $$

6. Again by Theorem 10 the matrix is

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}. $$

9. The first part of the transformation leaves $\mathbf{e}_1$ unchanged and the second part sends $\mathbf{e}_1$ to the point $(0, -1)$, which gives us the first column of the matrix. Now, the first part of the transformation

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}. $$
29. \[
\begin{bmatrix}
\begin{array}{ccc}
\ast & \ast & \ast \\
0 & \ast & 0 \\
0 & 0 & \ast \\
0 & 0 & 0
\end{array}
\end{bmatrix}
\]

30. \[
\begin{bmatrix}
\begin{array}{ccc}
\ast & \ast & \ast \\
0 & \ast & \ast \\
0 & 0 & \ast \\
0 & 0 & 0
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{ccc}
\ast & \ast & \ast \\
0 & 0 & \ast \\
0 & 0 & \ast \\
0 & 0 & 0
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{ccc}
\ast & \ast & \ast \\
0 & 0 & \ast \\
0 & 0 & \ast \\
0 & 0 & 0
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{ccc}
\ast & \ast & \ast \\
0 & \ast & \ast \\
0 & 0 & \ast \\
0 & 0 & 0
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{ccc}
\ast & \ast & \ast \\
0 & \ast & 0 \\
0 & 0 & \ast \\
0 & 0 & 0
\end{array}
\end{bmatrix}
\]

23. (a) True. Since the set of vectors \(\{e_1, \ldots, e_n\}\) (which are the columns of the identity matrix) spans \(\mathbb{R}^n\), every \(x \in \mathbb{R}\) can be written as a linear combination of them, and hence \(T(x)\) is determined by the \(T(e_i)\).

(b) True. It is easy to verify that \(T\) distributes sums and multiplication by scalars.

(c) False. Check proof in exercise 36.

(d) False. Since \(T\) is a function it will always be true that some vector in the domain maps onto some vector in the codomain. However that has nothing to do with the range being all \(\mathbb{R}^m\).

(e) True. The codomain of this linear transformation is \(\mathbb{R}^3\). However, the range of the transformation is the span of the columns of \(A\). Since \(A\) only has two columns, their span cannot be \(\mathbb{R}^3\).

33. By the definition of matrix product, the \(i\)-th column of \(B\) is precisely the vector \(Be_i\), which, by construction of \(T\) turns out to be \(T(e_i)\), which in turn, by definition, is the \(i\)-th column of \(A\).

36. Take two arbitrary vectors \(u, v \in \mathbb{R}^p\) and a scalar \(c \in \mathbb{R}\).

- By linearity of \(S\) we have \(S(cx) = cS(x)\), and in the same way \(T(cS(x)) = cT(S(x))\).
- By linearity of \(S\) we have \(S(u + v) = S(u) + S(v)\). And hence, using the linearity of \(T\) we get
  \[
  T(S(u + v)) = T(S(u) + v) = T(S(u)) + T(S(v)).
  \]

Chapter 2.1: 1,10,12,15,18,22,23,31,32.

1.
\[
-2A = \begin{bmatrix}
-2 & 0 & 2 \\
-8 & 10 & -4
\end{bmatrix}, \quad B - 2A = \begin{bmatrix}
5 & -5 & 3 \\
-7 & 6 & -7
\end{bmatrix}, \quad AC \text{ is undefined}, \quad CD = \begin{bmatrix}
1 & 13 \\
-5 & -6
\end{bmatrix}
\]

10. Indeed, both products yield the matrix \(\begin{bmatrix} 8 & -7 \\ -2 & 14 \end{bmatrix}\).

12. One possibility is \(B = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\).

15. (a) False. It is not even clear what \(a_1a_2\) means.
(b) False. The statement would be true if we were talking about \(BA\).
(c) True. By Theorem 2 in this chapter.
(d) True. By Theorem 3.
(e) False. The correct formula is \((AB)^T = B^T A^T\).

18. By the way in which the matrix product is defined, the first two columns of \(AB\) will also be equal.

22. Since the columns of \(B\) are linearly dependent there is an \(x \neq 0\) such that \(Bx = 0\). Hence \((AB)x = A(Bx) = A0 = 0\). Which implies that the columns of \(B\) are also linearly dependent.

23. It is enough to show that if \(x \neq 0\) then \(Ax \neq 0\). To see this take any \(x \neq 0\) and note that \(x = Ix = CAx = C(Ax)\) so \(C(Ax) \neq 0\) and hence \(Ax \neq 0\).

Now, if \(A\) had more columns than rows it would be possible to find a non-trivial solution to \(Ax = 0\), which we saw it is not possible.

31, 32 This follow directly from the definition of matrix product.

**Chapter 2.2: 10, 16, 20, 24, 30, 32.**

10. (a) False. If \(A\) and \(B\) are invertible the inverse of \(AB\) is \(B^{-1}A^{-1}\).

(b) False. This does not happen in most of the case, and example is \(A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\).

(c) True. The condition \(ad = bc\) implies that the columns of \(A\) are linearly dependent and hence the range of \(A\) is not all \(\mathbb{R}^2\), so \(A\) is not onto, and hence not invertible.

(d) True. Actually, the Gauss-Jordan procedure for reducing the matrix yields the inverse matrix.

(e) False. Again the matrix \(A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\) is a counterexample.

16. Let \(C = AB\), then \(A = CB^{-1}\) so \(A^{-1} = BC^{-1}\).

20. \(B = X(A - AX)^{-1}\) and both \(X\) and \((A - AX)^{-1}\) are invertible. Since products of invertible matrices is invertible, \(B\) is invertible.

24. Since \(Ax = b\) has a solution for every \(b\), the range of \(A\) is \(\mathbb{R}^n\) and hence the columns of \(A\) span \(\mathbb{R}^n\). But since \(A\) has \(n\) columns, in order to span \(\mathbb{R}^n\) they must be linearly independent, and hence \(A\) is one-to-one. So \(A\) is invertible.

30. The algorithm yields the matrix \(\begin{bmatrix} -\frac{7}{4} & 2 \\ \frac{5}{3} & -1 \end{bmatrix}\).

32. Since the columns satisfy teh following
\[
\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ -7 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = 0
\]
the columns are linearly dependent, and hence the matrix is invertible.
Chapter 2.3: 2, 5, 12, 15, 21, 28, 36.

2. Clearly one column is a multiple of the other, hence the matrix is not invertible.

5. Note that
\[
-2 \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 3 \\ 0 \\ -9 \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix} = 0
\]

the columns of the matrix are linearly dependent and hence the matrix is not invertible.

12. (a) True. By Theorem 8.
(b) True. Any \(n\)-linearly independent vectors span \(\mathbb{R}^n\).
(c) True. By Theorem 8.
(d) True. The condition implies that \(A\) is invertible and every invertible matrix is row equivalent to the identity matrix.
(e) True. Since \(A\) is \(n \times n\), then \(A\) is into if and only if it is onto.

15. No it cannot. If two columns are equal, then the columns of the matrix are linearly dependent, and hence the matrix is not invertible.

21. Yes. If \(G\) is rectangular, with more columns than rows, \(G\) can be onto but not into.

28. Proceed by contradiction. Suppose that \(B\) is not invertible, hence the columns of \(B\) are linearly dependent. In 2.1.22 we showed that if the columns of \(B\) are linearly dependent, then the columns of \(AB\) are also linearly dependent, and hence \(AB\) is not invertible, which is a contradiction.

36. Let \(A\) be the matrix associated to \(T\). Then \(A\) is invertible. It is easy to verify that if we define \(T^{-1}(x) = A^{-1}x\) then \(T^{-1}\) is the inverse function of \(T\). It is also clear that \(T^{-1}\) is also one-to-one.