Math 54 Final Exam, Prof. Srivastava
December 14, 2018, 3:10pm–6:00pm, 150 Wheeler (170 minutes).

Name: ______________________________

SID: ______________________________

Instructions: Write all answers in the provided space. This exam includes three scratch spaces which will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Question:	1	2	Total
Points:	24	24	48

Do not turn over this page until your instructor tells you to do so.

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1. (24 points) Fill in (T) (always true) or (F) (sometimes false) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If A is an $m \times n$ matrix, then the number of pivots in the REF of A is equal to the number of nonzero singular values of A. (T) (F)

   Solution: True. Both are equal to the rank of A.

   (b) If V, W, Y are finite dimensional vector spaces and $T : V \to W$ and $S : W \to Y$ are isomorphisms, then V and Y must have the same dimension. (T) (F)

   Solution: True. Isomorphisms preserve dimension, so $\dim(V) = \dim(W) = \dim(Y)$.

   (c) If $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ are orthonormal bases of $\mathbb{R}^2$, then the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ has the property that $\|P_{\mathcal{C}\leftarrow\mathcal{B}}x\|^2 = \|x\|^2$ for every $x \in \mathbb{R}^2$.

   Solution: True. If E is the standard basis then $P_{C\leftarrow B} = P_{C\leftarrow E}P_{E\leftarrow B} = P_{E\leftarrow C}^{-1}P_{E\leftarrow B}$. Since $P_{E\leftarrow B}$ and $P_{E\leftarrow C}$ contain the columns of B and C they are orthogonal matrices. Since inverses and products of orthogonal matrices are orthogonal, the conclusion follows.

   (d) For every nonzero vector $v \in \mathbb{R}^3$, there is a basis of $\mathbb{R}^3$ containing v. (T) (F)

   Solution: True. Just add a vector $w \notin \text{span}\{v\}$ and then a vector $z \notin \text{span}\{v, w\}$ to get a basis.

   (e) Every nonzero vector $v \in \mathbb{R}^2$ can be written as $v = a + b$ for some nonzero vectors a, b with $a \cdot b = 0$. (T) (F)

   Solution: True. Choose any unit vector u such that u is not parallel or orthogonal to v, and let $a = (v \cdot u)u$ and $b = v - a$. By the decomposition theorem these vectors are necessarily orthogonal, and nonzero by the way we chose u.

   (f) If V is a finite dimensional vector space and $S = \{v_1, \ldots, v_k\}$ spans V, then S contains a basis of V as a subset. (T) (F)

   Solution: True. If S is already a basis we are done, otherwise it is not linearly independent and we can remove some vector which is in the span of the previous ones. Repeat this process until you get a linearly independent set. Since we didn't change the span, this set is a basis.

(g) If x is a solution to $Ax = b$ then $x \in \text{Row}(A)$. (T) (F)

Solution: False. If x is a solution to $Ax = b$ then $A(x + z) = Ax + 0 = b$ for any $z \in \text{Nul}(A)$, but $x + z \notin \text{Row}(A)$.

(h) If W is a subspace of $\mathbb{R}^n$ then the standard matrix of the orthogonal projection onto W must be diagonalizable. (T) (F)

Solution: True. If W has dimension k and U is an $n \times k$ matrix containing an orthonormal basis of W as its columns then the standard matrix of the projection is UU^T. Since $(UU^T)^T = UU^T$, the spectral theorem implies that this is diagonalizable.

(i) If $A = A^T$ and P is the orthogonal projection onto the column space of A, then $AP = PA$. (T) (F)

Solution: True. Assume A is $n \times n$. By the spectral theorem A is orthogonally diagonalizable, so let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A, arranged so that $Au_i = \lambda_i u_i$ with $\lambda_i \neq 0$ for $i \leq k$ and $Au_i = 0$ for $i > k$. Observe that $u_1 \ldots, u_k \in \text{Col}(A)$ by the eigenvector equation, so for these vectors we have

$$APu_i = Au_i$$

and

$$PAu_i = P(\lambda_i u_i) = \lambda_i u_i = Au_i.$$

For u_i with $i > k$ we have

$$APu_i = A(0) = 0 = Au_i$$

since $u_i \in \text{Col}(A)^\perp = \text{Row}(A)^\perp = \text{Nul}(A)$ since $\text{Col}(A) = \text{Row}(A)$ because A is symmetric. Similarly, for these vectors

$$PAu_i = P(0) = 0 = Au_i.$$

Altogether, we conclude that $APu_i = PAu_i = Au_i$ for all of the basis vectors $u_1, \ldots, u_n$, which means that $AP = PA = A$.

(j) If $A = A^T$ and all the eigenvalues of A are positive, then there is a matrix B such that $B^2 = A$. (T) (F)

Solution: True. By the spectral theorem we have $A = UDU^T$ for some orthogonal U and diagonal D which has positive entries $\lambda_1, \ldots, \lambda_n > 0$ Let C be the diagonal matrix with entries $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$, which are real since the eigenvalues were positive, and note that $C^2 = D$. Letting $B = UCU^T$ we see that

$$(UCU^T)^2 = UC(U^TU)CU^T = UC^2U^T = UDU^T = A,$$

as desired.

(k) If A is invertible and σ is a singular value of A then $1/\sigma$ is a singular value of A^{-1}.
T F

Solution: True. The SVD implies that $A = U\Sigma V^T$ for orthogonal U, V and diagonal Σ, which must be square since an invertible matrix has to be square, and which has positive diagonal entries. Now

$$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T,$$

since U, V are orthogonal. But this is just the SVD of A^{-1}, so A must have singular values equal to the entries of Σ^{-1} which are the reciprocals of the singular values of A.

(l) If A and B are similar square matrices, then every singular value of A is also a singular value of B.
T F

Solution: False. You should be suspicious because the singular values of A tell us how much A can stretch or shrink a vector, and this can be changed a lot by a similarity transformation $B = PAP^{-1}$ when P is not orthogonal. In particular, a square matrix A is orthogonal if and only if all of its singular values are equal to one. Taking $A = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ to be counterclockwise rotation by 45 degrees and $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ we find that $B = PAP^{-1}$ is not orthogonal, so it has singular values other than one, which means that similarity does not preserve singular values.

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (4 points) A 2×3 real matrix A such that $A^T A$ and AA^T are both invertible.

   Solution: Does not exist. The reason is that $\text{rank}(A^T A) \leq \text{rank}(A^T)$ since $\text{Col}(A^T A) \subseteq \text{Col}(A^T)$, but $\dim(\text{Col}(A^T)) = \text{rank}(A^T) \leq 2$ since A^T is 3×2, so $A^T A$ is a 3×3 matrix with rank at most 2 and cannot be invertible.

   (b) (4 points) A subspace $W \subseteq \mathbb{R}^3$ such that $\det(P) \neq 0$ and $\det(I - P) \neq 0$, where P is the standard matrix of the orthogonal projection onto W.

   Solution: Does not exist. Conceptual proof: Observe that $I - P$ is the projection onto $W^\perp$, so $\text{rank}(P) + \text{rank}(I - P) = \text{rank}(W) + \text{rank}(W^\perp) = 3$. Thus one of these matrices must have rank less than 3, which means its determinant must be zero.

   Algebraic proof: $\det(P)\det(I - P) = \det(P(I - P)) = \det(P - P^2) = \det(P - P) = 0$, where $P^2 = P$ since P is a projection.

   (c) (4 points) A real 2×2 orthogonal matrix with all eigenvalues real and strictly negative.

   Solution: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(d) (4 points) A second order differential operator $T : F \to F$ of the form

$$T(y) = y'' + by'$$

with infinitely many eigenvalues, where $b \in \mathbb{R}$ and F is the vector space of infinitely differentiable functions $y : \mathbb{R} \to \mathbb{R}$.

Solution: The eigenvalue equation in this context is:

$$T(y) = y'' + by' = \lambda y,$$

so the question is asking whether there is a b such that this is satisfied for infinitely many λ and nonzero functions y. But this is just a second order ODE, which always has at least one real solution. So actually every $\lambda \in \mathbb{R}$ is an eigenvalue, regardless of the value of b. For a concrete example we may take $b = 0$.

(e) (4 points) A second order differential equation $y''(t) + by'(t) + cy(t) = 0$ which has among its solutions $y_1(t) = e^t(\cos(3t) - 3\sin(3t))$ and $y_2(t) = e^t(3\cos(3t) - \sin(3t))$.

Solution: This is a 'reverse engineering question'. Observe that any homogeneous ODE with solutions containing $e^t \cos(3t)$ and $e^t \sin(3t)$ will have y_1 and y_2 among its solutions by taking linear combinations. The auxiliary polynomial $r^2 + br + c$ of such an ODE must have roots $1 \pm 3i$, so we have

$$r^2 + br^2 + c = (r - (1 + 3i))(r - (1 - 3i)) = r^2 - 2r + 10.$$

Thus the answer is $y'' - 2y' + 10y = 0$.

(f) (4 points) An inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ such that $\langle v, w \rangle > 0$ for all pairs of nonzero vectors $v, w \in \mathbb{R}^3$.

Solution: Does not exist. For any nonzero vector v and $w = -v$, we have $\langle v, -v \rangle = -\langle v, v \rangle = -\|v\|^2 < 0$ by the properties of inner products.

Name and SID: Estella Chen [redacted]

3. (7 points) Suppose V is an infinite dimensional vector space, $v_1, v_2, v_3 \in V$ are vectors, and $T : V \to \mathbb{R}^2$ is a linear transformation such that such that

$$T(v_1 + 2v_2 + 3v_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(3v_1 + 2v_2 + v_3) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$-1 \quad -2$
$1 \quad 2$

What is the value of $T(5v_1 + 2v_2 - v_3)$?

$-1 \quad 6 \quad 5$
$-2 \quad 4 \quad 2$
$-3 \quad 2 \quad -1$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & 2 \\ 3 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -4 & -8 \\ 0 & -8 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T(5v_1+2v_2-v_3) = -1(T(v_1+2v_2+3v_3)) + 2(T(3v_1+2v_2+v_3))$$

$$= -\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 3 \end{bmatrix}}$$

4. (7 points) Let M_4 be the vector space of real 4×4 matrices with entrywise addition and scalar multiplication, and consider the linear transformation $T : M_4 \to M_4$ defined by $T(X) = AX$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ h & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

For which values of h is T onto? Explain your reasoning.

Let the columns of X be x_1, x_2, x_3, x_4.
This is like mapping x_1, x_2, x_3, x_4 to b_1, b_2, b_3, b_4 such that we have all combinations of b_1, b_2, b_3, b_4. This means $Ax_1 = b_1$, $Ax_2 = b_2$, $Ax_3 = b_3$, $Ax_4 = b_4$ have solutions for all b_1, b_2, b_3, b_4, which just means that the columns of A span $\mathbb{R}^4$. Since A is 4×4, it must have 4 pivots.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ h & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ h & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so $\boxed{h \neq 1}$

$Ax = b$ has sol. for all b.

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5. (7 points) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad , v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Find vectors $x \in \text{Row}(A)$ and $y \in \text{Nul}(A)$ such that $v = x + y$.

ROW(A) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$V_2 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0(1)+1(1)+1(0)}{1^2+1^2+0^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$

$V_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$\text{proj}_{v_1} V = \frac{1(1)+0(1)+1(0)}{1^2+1^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$

$\text{proj}_{v_2} V = \frac{1(-1)+0(1)+1(2)}{1^2+1^2+2^2} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/6 \\ 1/3 \end{bmatrix}$

$x = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$y = v - x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

$$\boxed{x = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} \quad y = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}}$$

[Scratch Space 1]

Name and SID: Estella Chen [redacted]

6. (7 points) Find the singular value decomposition $A = U\Sigma V^T$ (i.e., where U and V are orthogonal and Σ is nonnegative diagonal) of the matrix

$AV = U\Sigma$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

$A^TA = V\Sigma^2V^T = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(A^TA - \lambda I) = (4-\lambda)(1-\lambda) = 0$

$\lambda = 4, 1$ $\qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\sigma = 2, 1$

$\lambda = 4 \quad \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow$ eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda = 1 \quad \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Av_1 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Av_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\boxed{\underset{A}{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{\Sigma}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}\underset{V^T}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}$$

[Scratch Space 2]

Name and SID: Estella Chen [redacted]

7. (a) (3 points) Find a basis for the space of real solutions to the homogeneous differential equation:

$$y''(t) - 9y'(t) + 20y = 0.$$

$r^2 - 9r + 20 = 0$

$(r-4)(r-5) = 0$

$y = c_1 e^{4t} + c_2 e^{5t}$

$\boxed{\{e^{4t}, e^{5t}\}}$

(b) (2 points) Find a solution to the initial value problem for the equation in part (a) with initial data $y(0) = 0$ and $y'(0) = 0$.

$y(0) = c_1 + c_2 = 0$

$y' = 4c_1 e^{4t} + 5c_2 e^{5t}$

$y'(0) = 4c_1 + 5c_2 = 0.$

$4c_1 + 5c_2 = 0$

$4c_1 + 4c_2 = 0$

$c_2 = 0$

$c_1 = 0$

$\boxed{y(t) = 0}$

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(c) (3 points) Find the general solution to the inhomogeneous differential equation:

$$y''(t) - 9y'(t) + 20y = t.$$

Guess $y_p = at + b$.

$y_p' = a$

$y_p'' = 0$.

$-9a + 20(at + b) = t$

$-9a + 20at + 20b = t$.

$20a = 1 \qquad a = \frac{1}{20}$

$-9a + 20b = 0 \qquad 20b = \frac{9}{20}$

$b = \frac{9}{400}$

$y_p(t) = \frac{1}{20}t + \frac{9}{400}$

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{5t} + \frac{1}{20}t + \frac{9}{400}}$$

(d) (2 points) Find a solution to the initial value problem for the equation in part (c) with initial data $y(0) = 0$ and $y'(0) = 0$.

$y(0) = c_1 + c_2 + \frac{9}{400} = 0$

$\frac{1}{20}$

$y'(t) = 4c_1 + 5c_2 +$ [redacted] $= 0$

$\frac{1}{20}$

$4c_1 + 5c_2 +$ [redacted] $= 0$

$4c_1 + 4c_2 + \frac{36}{400} = 0$

$c_2 = \frac{36}{400}$ [redacted] $- \frac{20}{400} = \frac{16}{400} = \frac{1}{25}$

$c_1 = -c_2 - \frac{9}{400}$ [redacted] $= -\frac{25}{400} = -\frac{1}{16}$

$y(t) =$ [redacted] $e^{4t} +$ [redacted] $e^{5t} + \frac{1}{20}t + \frac{9}{400}$

$-1/16 \qquad 1/25$

Name and SID: Estella Chen [redacted]

8. Consider the homogeneous system of ODE:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t).$$

(a) (4 points) Find a basis for the vector space V of real solutions (i.e., differentiable functions $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ satisfying the equation) of this system.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \det(A - \lambda I) = (1-\lambda)(1-\lambda) - 1 = 0$

$\lambda^2 - 2\lambda + \cancel{1} \cancel{-1} = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0 \quad \lambda = 0, 2$

$\lambda = 0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x(t) = c_1 e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \right\}}$$

(b) (1 point) What is the dimension of V?

$\boxed{2}$ It has two vectors in its basis.

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(c) (2 points) Consider the linear transformation $T : V \to \mathbb{R}^2$ defined by

$$T(\mathrm{x}) = \mathrm{x}(0).$$

Is T one to one? Explain why or why not.

Let $x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x \in V$

Then $x(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

T is one-to-one because it has the transformation matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, which is invertible by inspection.

No two vectors $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ map to the same vector.

[Scratch Space 3]

Name and SID: Estella Chen [redacted]

9. Let F be the vector space of piecewise continuous functions with piecewise continuous derivative on $[-\pi, \pi]$.

(a) (5 points) Find the Fourier series of the function $f(x) = x$ defined on $[-\pi, \pi]$.

$$\frac{a_0}{2} = \frac{\int_{-\pi}^{\pi} x\,dx}{\int_{-\pi}^{\pi} 1\,dx} = 0 \text{ because } f \text{ is odd.}$$

$a_n = 0$ because f is odd.

$$b_n = \frac{\int_{-\pi}^{\pi} x\sin(nx)dx}{\pi} \qquad \begin{matrix} u = x & dv = \sin(nx)dx \\ du = dx & v = -\frac{1}{n}\cos(nx) \end{matrix}$$

$$= \frac{\left(\frac{-x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)dx\right)\Big|_{-\pi}^{\pi}}{\pi}$$

$$= \frac{\left(\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_{-\pi}^{\pi}}{\pi}$$

$$= \frac{\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\cdot 0 - \left(\frac{\pi}{n}\cos(-n\pi) + 0\right)}{\pi}$$

$$= \frac{-2}{n}\cos(n\pi) = \frac{-2}{n}(-1)^n$$

$$\boxed{\sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin(nx)}$$

(b) (2 points) Consider the four dimensional subspace $S = \text{span}\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ of F. Find the orthogonal projection of f onto the subspace S with respect to the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ on F.

$$\text{proj}_S f = \frac{\int_{-\pi}^{\pi} x\sin x\,dx}{\pi} + \cancel{\frac{\int_{-\pi}^{\pi} x\cos x\,dx}{\pi}} + \frac{\int_{-\pi}^{\pi} x\sin(2x)dx}{\pi} + \cancel{\frac{\int_{-\pi}^{\pi} \cos(2x)x\,dx}{\pi}}$$

f is odd   f is odd..

From the calculations in part (a).

$$\text{proj}_S f = \frac{-2}{1}(-1)^1 \sin x + \frac{-2}{2}(-1)^2 \sin(2x)$$

$$= \boxed{2\sin x - \sin(2x)}$$