MATH 54 FINAL EXAM, PROF. SRIVASTAVA
DECEMBER 14, 2018, 3:10PM–6:00PM, 150 WHEELER (170 MINUTES).

Name: ______________________________

SID: ______________________________

INSTRUCTIONS: Write all answers in the provided space. This exam includes three scratch spaces which will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Sign here: ______________________________

NAME OF THE STUDENT TO YOUR LEFT: ______________________________
NAME OF THE STUDENT TO YOUR RIGHT: ______________________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	24	24	7	7	7	7	10	7	7	100

Do not turn over this page until your instructor tells you to do so.

Name and SID: ____________________

1. (24 points) Fill in (T) (always true) or (F) (sometimes false) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If A is an $m \times n$ matrix, then the number of pivots in the REF of A is equal to the number of nonzero singular values of A. (T) (F)

   (b) If V, W, Y are finite dimensional vector spaces and $T : V \to W$ and $S : W \to Y$ are isomorphisms, then V and Y must have the same dimension. (T) (F)

   (c) If $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ are orthonormal bases of $\mathbb{R}^2$, then the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ has the property that $\|P_{\mathcal{C}\leftarrow\mathcal{B}}x\|^2 = \|x\|^2$ for every $x \in \mathbb{R}^2$. (T) (F)

   (d) For every nonzero vector $v \in \mathbb{R}^3$, there is a basis of $\mathbb{R}^3$ containing v. (T) (F)

   (e) Every nonzero vector $v \in \mathbb{R}^2$ can be written as $v = a + b$ for some nonzero vectors a, b with $a \cdot b = 0$. (T) (F)

   (f) If V is a finite dimensional vector space and $S = \{v_1, \ldots, v_k\}$ spans V, then S contains a basis of V as a subset. (T) (F)

   (g) If x is a solution to $Ax = b$ then $x \in \text{Row}(A)$. (T) (F)

   (h) If W is a subspace of $\mathbb{R}^n$ then the standard matrix of the orthogonal projection onto W must be diagonalizable. (T) (F)

   (i) If $A = A^T$ and P is the orthogonal projection onto the column space of A, then $AP = PA$. (T) (F)

   (j) If $A = A^T$ and all the eigenvalues of A are positive, then there is a matrix B such that $B^2 = A$. (T) (F)

   (k) If A is invertible and σ is a singular value of A then $1/\sigma$ is a singular value of A^{-1}. (T) (F)

   (l) If A and B are similar square matrices, then every singular value of A is also a singular value of B. (T) (F)

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (4 points) A 2×3 real matrix A such that $A^T A$ and AA^T are both invertible.

   (b) (4 points) A subspace $W \subseteq \mathbb{R}^3$ such that $\det(P) \neq 0$ and $\det(I - P) \neq 0$, where P is the standard matrix of the orthogonal projection onto W.

   (c) (4 points) A real 2×2 orthogonal matrix with all eigenvalues real and strictly negative.

(d) (4 points) A second order differential operator $T : F \to F$ of the form

$$T(y) = y'' + by'$$

with infinitely many eigenvalues, where $b \in \mathbb{R}$ and F is the vector space of infinitely differentiable functions $y : \mathbb{R} \to \mathbb{R}$.

(e) (4 points) A second order differential equation $y''(t) + by'(t) + cy(t) = 0$ which has among its solutions $y_1(t) = e^t(\cos(3t) - 3\sin(3t))$ and $y_2(t) = e^t(3\cos(3t) - \sin(3t))$.

(f) (4 points) An inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ such that $\langle v, w \rangle > 0$ for all pairs of nonzero vectors $v, w \in \mathbb{R}^3$.

3. (7 points) Suppose V is an infinite dimensional vector space, $v_1, v_2, v_3 \in V$ are vectors, and $T : V \to \mathbb{R}^2$ is a linear transformation such that such that

$$T(v_1 + 2v_2 + 3v_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(3v_1 + 2v_2 + v_3) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

What is the value of $T(5v_1 + 2v_2 - v_1)$?

4. (7 points) Let M_4 be the vector space of real 4×4 matrices with entrywise addition and scalar multiplication, and consider the linear transformation $T : M_4 \to M_4$ defined by $T(X) = AX$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ h & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

For which values of h is T onto? Explain your reasoning.

5. (7 points) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad , v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Find vectors $x \in \mathrm{Row}(A)$ and $y \in \mathrm{Nul}(A)$ such that $v = x + y$.

[Scratch Space 1]

Name and SID: ____________________

6. (7 points) Find the singular value decomposition $A = U\Sigma V^T$ (i.e., where U and V are orthogonal and Σ is nonnegative diagonal) of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

[Scratch Space 2]

7. (a) (3 points) Find a basis for the space of real solutions to the homogeneous differential equation:

$$y''(t) - 9y'(t) + 20y = 0.$$

(b) (2 points) Find a solution to the initial value problem for the equation in part (a) with initial data $y(0) = 0$ and $y'(0) = 0$.

(c) (3 points) Find the general solution to the inhomogeneous differential equation:

$$y''(t) - 9y'(t) + 20y = t.$$

(d) (2 points) Find a solution to the initial value problem for the equation in part (c) with initial data $y(0) = 0$ and $y'(0) = 0$.

8. Consider the homogeneous system of ODE:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t).$$

(a) (4 points) Find a basis for the vector space V of real solutions (i.e., differentiable functions $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ satisfying the equation) of this system.

(b) (1 point) What is the dimension of V?

(c) (2 points) Consider the linear transformation $T : V \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \mathbf{x}(0).$$

Is T one to one? Explain why or why not.

[Scratch Space 3]

9. Let F be the vector space of piecewise continuous functions with piecewise continuous derivative on $[-\pi, \pi]$.

   (a) (5 points) Find the Fourier series of the function $f(x) = x$ defined on $[-\pi, \pi]$.

   (b) (2 points) Consider the four dimensional subspace $S = \text{span}\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ of F. Find the orthogonal projection of f onto the subspace S with respect to the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ on F.