Math 54 Fall 2016 Practice Midterm 2

Nikhil Srivastava

50 minutes, closed book, closed notes

1. True or False (no need for justification):

   (a) If V is a vector space with a finite basis then V is isomorphic to $\mathbb{R}^n$ for some n.

   (b) The system $A^T Ax = A^T b$ is consistent for all A and b.

   (c) If A and B are similar and A is diagonalizable then B must be diagonalizable.

   (d) The rank of a square matrix is equal to the number of nonzero eigenvalues (counted with multiplicity).

   (e) If x and y are arbitrary nonzero vectors in $\mathbb{R}^n$ then there is a basis B of $\mathbb{R}^n$ such that $[x]_B = y$.

   (f) Every eigenvalue of a square matrix A is a pivot of A in the reduced row echelon form of A.

   (g) If A is a square matrix then A and A^T have the same eigenvalues.

   (h) An upper triangular matrix is always diagonalizable.

   (i) An set of orthogonal vectors is always linearly independent.

   (j) If $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent and W is a subspace of $\mathbb{R}^n$ then $\mathrm{Proj}_W(v_1), \dots, \mathrm{Proj}_W(v_k)$ must also be linearly independent.

2. Let $V = \mathbb{R}^{3\times 3}$ denote the vector space of real 3×3 matrices with addition and scalar multiplication defined entrywise. Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   and consider the subset

$$W = \{X \in V : XM = MX\}$$

   of matrices in V which commute with M. Is W a subspace of V? If so, prove it. If not, explain why.

3. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2\}$ be the vector space of polynomials of degree at most 2 with coefficient-wise operations, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by
$$T(q) = \frac{d^2}{dt^2} q + t \cdot \frac{d}{dt} q + 3q.$$
Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of T with respect to $\mathcal{B}$ is diagonal? If so, find such a basis as well as the corresponding matrix. If not, explain why.

4. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Find the eigenvalues of A. Compute A^{11}.

5. Consider the vectors
$$x_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 12 \\ -4 \\ 7 \\ 1 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix},$$
and let $W = \text{Span}\{x_1, x_2\}$. Find vectors w and z such that $w \in W, z \in W^\perp$, and $y = w + z$. What is the distance between y and the closest point in W to y?

6. Let W be a subspace of $\mathbb{R}^n$ and let P be the standard matrix of the projection onto W, i.e., $P = [\text{Proj}_W]$ where $\text{Proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation which projects onto W. (a) Show that $P^2 = P$. (b) Use this to show that the eigenvalues of P are all either 0 or 1. (c) What is the eigenspace corresponding to the eigenvalue 1 of P?