Math 54 Fall 2016 Practice Midterm 1

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closed book, closed notes

1. True or False:

   (a) If the reduced row echelon form of the augmented matrix of a linear system has a column containing only zeros, then it must be consistent.

   False. A column of zeros does not say anything about a pivot in the augmented column, which is the relevant test for consistency.

   (b) If the columns of A are linearly independent, then $Ax = b$ is consistent for every b.

   False. Linear independence of the columns implies uniqueness but not existence.

   (c) If A has linearly dependent columns, then $Ax = 0$ is has infinitely many solutions.

   True. If A has linearly dependent cols there is a nontrivial solution, so there must be infinitely many solutions.

   (d) If A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, then the set of solutions to $Ax = b$ is a linear subspace of $\mathbb{R}^n$.

   False. This is true for $b = 0$ but not in general. This may be verified algebraically. Geometrically, the solution set of $Ax = b$ is a *translation* of a plane through the origin.

   (e) If a linear subspace of $\mathbb{R}^n$ contains more than one vector, then it must contain infinitely many vectors.

   True. If it has more than one vector it must have a nonzero vector, and all scalar mutliples of this vector must also be in the subspace.

   (f) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation then T cannot be both $1-1$ and onto.

   True. In particular, the columns of the standard matrix of T cannot be linearly independent, since there are three columns in $\mathbb{R}^2$, so T cannot be $1-1$.

   (g) If A is an $m \times n$ matrix and B is an $n \times p$ matrix whose first column contains only zeros, then the first column of AB also contains only zeros.

   True. By the columnwise definition of matrix vector multiplication, the first column of B is b_1 then the first column of AB is Ab_1.

(h) If A, B, C are invertible then the product ABC must also be invertible.

True. The inverse is $C^{-1}B^{-1}A^{-1}$.

(i) If A is a square matrix such that $A^2 = 0$ then $A = 0$.

False. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(j) If A and B are square matrices of rank 2 then the product AB has rank at most 2.

True. The rank of a matrix is the dimension of the span of its columns, which is the maximum number of linearly independent columns. Assume A is $m \times n$ and B is $n \times p$. Let $b_1, \ldots, b_p$ be the columns of B. Since B has rank 2, every subset of three columns of B is linearly dependent. Since the columns of AB are $Ab_1, \ldots, Ab_p$, every subset of three columns of AB is also linearly dependent, since multiplication by A preserves linear dependencies. Thus, AB has rank at most 2.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find a 3×2 matrix B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: Letting b_1 and b_2 denote the (unknown) columns of B, we observe that this is equivalent to solving the matrix equations:

$$Ab_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad Ab_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which have augmented matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Applying row reduction we obtain the RREF:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

so both systems have infinitely many solutions, given by

$$b_1 = \begin{bmatrix} 1 - x_3 \\ 0 \\ x_3 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 - x_3 \\ 1 \\ x_3 \end{bmatrix},$$

where x_3 is a free variable. Thus, there are infinitely many possibilities for B, given by

$$B = \begin{bmatrix} 1 - x_3 & -1 - x_3 \\ 0 & 1 \\ x_3 & x_3 \end{bmatrix},$$

and a particular B can be found by plugging in (for instance) $x_3 = 0$.

3. Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 6 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, v_5 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 6 \end{bmatrix}.$$

(a) Let $H = \text{span}\{v_1, v_2, v_3, v_4, v_5\}$. Find a subset of the given vectors which forms a basis for H.

(b) What is the dimension of H?

(c) Determine whether the vector

$$w = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$

lies in H.

Solution:

(a) The desired subspace is the column space of the matrix with the v_i as columns, namely:

$$A = \begin{bmatrix} 1 & 1 & 3 & 0 & 2 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 0 & 3 & 0 & 3 \\ 4 & 1 & 6 & 1 & 6 \end{bmatrix}.$$

Recall that a basis for the column space is given by the columns of A corresponding to the pivot columns of the REF of A. Applying row reduction (details omitted), we obtain the REF:

$$R = \begin{bmatrix} 1 & 1 & 3 & 0 & 2 \\ 0 & -3 & -6 & 1 & -2 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first, second, and fourth columns are pivot columns, a basis for the column space is given by $\{v_1, v_2, v_4\}$.

(b) The dimension of H is three, since it has a basis with three vectors.

(c) To test whether w is in $H = \text{span}\{v_1, v_2, v_4\}$, we must solve the linear system with augmented matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 0 \\ 3 & 0 & 0 & 3 \\ 4 & 1 & 1 & 4 \end{bmatrix}.$$

The REF of this matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This system is consistent, so indeed w is in the subspace.

4. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 + x_3 \end{bmatrix},$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates a vector about the origin by $\pi/4$ radians counterclockwise.

(a) Determine the standard matrix of the composition $T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by applying T_1 and then T_2.

(b) Determine whether $T_2 \circ T_1$ is onto.

(c) Determine whether $T_2 \circ T_1$ is one to one.

Solution:

(a) To find the standard matrix of $T = T_2 \circ T_1$, we apply it to the standard basis of $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

where the first arrows indicate application of $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and the second indicate $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$. The standard matrix is the 2×3 matrix with $T(e_1), T(e_2), T(e_3)$ as columns, namely:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) To determine whether T is onto, we recall that this is equivalent to the columns of its standard matrix spanning $\mathbb{R}^2$, which is equivalent to having a pivot in every row. Subtracting the first row from the second, we find that a REF of A is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 & 1\\ 0 & 2 & 2\end{bmatrix},$$

which has this property, so T must be onto.

(c) Recall that T is one to one if and only if the columns of its standard matrix are linearly independent. Since 3 vectors in $\mathbb{R}^2$ cannot be linearly independent, we conclude that T is not one to one.

5. (a) Give an example of a 3×3 matrix whose null space has dimension 1.

   (b) Give an example of a 3×3 matrix whose column space has dimension 1.

   (c) Is there a 3×3 matrix whose null space and column space *both* have dimension 1?

Solution:

(a) The dimension of the null space of a matrix is equal to the number of free variables in its REF. So the matrix

$$\begin{bmatrix}1 & 0 & *\\ 0 & 1 & *\\ 0 & 0 & 0\end{bmatrix}$$

has a null space of dimension one, where $*$ indicates any number.

(b) The dimension of the column space of a matrix is the number of pivot columns. So any matrix of type

$$\begin{bmatrix}1 & * & *\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

has a column space of dimension one.

(c) No, there is no such matrix, because the dimension theorem tells us that the sum of the dimensions of the row space and column space is equal to 3 for any matrix with 3 columns, and we have $1 + 1 \neq 3$.