Math 54 Second Midterm Exam, Prof. Srivastava
October 31, 2016, 4:10pm–5:00pm, 155 Dwinelle Hall.

Name: ____________________

SID: ____________________

Instructions: Write all answers in the provided space. This exam includes one page of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Name of the student to your right: ____________________

Question	Points
1	20
2	15
3	10
4	7
5	8
Total:	60

Do not turn over this page until your instructor tells you to do so.

Name and SID: ______________________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If A is a square matrix then $\mathrm{Col}(A) \cap \mathrm{Null}(A) = \{0\}$. **T** **F**

   > **Solution:** False. First of all, you should be suspicious of such a claim because for a rectangular $m \times n$ matrix $\mathrm{Col}(A) \leq \mathbb{R}^n$ and $\mathrm{Null}(A) \leq \mathbb{R}^m$, which means that they don't even live in the same vector space, and shouldn't have any particular relation in general.
   >
   > For a concrete example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for which $\mathrm{Col}(A) = \mathrm{Null}(A) = \mathrm{Span}\{e_1\}$.

   (b) If $v \in \mathrm{Span}\{w, a\}$ and $w \in \mathrm{Span}\{a, b\}$ then $v \in \mathrm{Span}\{a, b\}$, where v, w, a, b are vectors in a vector space. **T** **F**

   > **Solution:** True. If $v = c_1 w + c_2 a$ and $w = d_1 a + d_2 b$ then $v = c_1 d_1 a + c_1 d_2 b + d_2 b$.

   (c) If V and W are $n-$dimensional vector spaces then there is a linear transformation $T : V \to W$ which is both 1-1 and onto. **T** **F**

   > **Solution:** True. There are several ways to see this. Let $B = \{b_1, \ldots, b_n\}$ be a basis for V and let $C = \{c_1, \ldots, c_n\}$ be a basis for W. Then the coordinate mappings $P_B^{-1} : V \to \mathbb{R}^n$ and $P_C^{-1} : W \to \mathbb{R}^n$ are isomorphisms (which means they are 1-1 and onto). The latter has an inverse, $P_C : \mathbb{R}^n \to W$, which is also an isomorphism. The composition $P_C \circ P_B^{-1} : V \to W$ is thus also an isomorphism, as desired.
   >
   > Another, direct way is to consider the linear transformation $T : V \to W$ defined by $T(b_i) = c_i$ an show that its matrix is invertible (which will follow from B and C being bases).

   (d) If V is an $n-$dimensional vector space then every linearly independent subset of V contains at most n vectors. **T** **F**

   > **Solution:** True. This is one of the theorems in the book.

   (e) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, $[T]$ is the standard matrix of T, and $[T]_{\mathcal{B}}$ is the matrix of T with respect to another basis $\mathcal{B}$, then $\det([T]) = \det([T]_{\mathcal{B}})$. **T** **F**

   > **Solution:** True. Let E be the standard basis. We have
   >
   > $$[T]_B = P_{\mathcal{B} \leftarrow E}[T]P_{E \leftarrow \mathcal{B}} = P[T]P^{-1}$$

for $P = P_{\mathcal{B}\leftarrow E}$. Since $\det(XY) = \det(X)\det(Y)$ and $\det(X^{-1}) = 1/\det(X)$, we have

$$\det([T]_B) = \det(P)\det([T])\det(P^{-1}) = \det(P)\det([T])\frac{1}{\det(P)} = \det([T]).$$

(f) If λ is an eigenvalue of A and μ is an eigenvalue of B then $\lambda + \mu$ is an eigenvalue of $A + B$. **T F**

Solution: False. This was one of the hardest questions. You should be suspicious just by considering what the definitions say: if λ and μ are eigenvalues of A and B then we know that $Ax = \lambda x$ and $By = \mu y$ for some eigenvectors $x, y \neq 0$. Adding these equations gives

$$Ax + By = \lambda x + \mu y,$$

which would only imply the statement in question if x and y are scalar multipes of each other. So to find a counterexample, you should look for two matrices with *different* eigenvectors.

Most pairs of matrices have this property. Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, each with eigenvalues equal to zero. However, the sum $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues equal to ± 1.

(g) If A is a 4×4 real matrix with two distinct real eigenvalues and two complex eigenvalues, then it must be diagonalizable over $\mathbb{C}$. **T F**

Solution: True. Since A is real its complex eigenvalues must be conjugates of each other, i.e., $a + ib$ and $a - ib$ for some $b \neq 0$. Thus, A has four distinct eigenvalues, which implies that it must be diagonalizable.

(h) If W is a subspace of $\mathbb{R}^n$ and x is a vector in $\mathbb{R}^n$ such that $\text{Proj}_W(x) = 0$ then $x \in W^\perp$. **T F**

Solution: True. By the unique decomposition theorem, we have $x = y + z$ for $y = \text{Proj}_W(x)$ and $z \in W^\perp$. Since $y = 0$ we have $x \in W^\perp$.

(i) Let W be a subspace of $\mathbb{R}^n$ and let $P = [\text{Proj}_W]$ be the $n \times n$ matrix of the orthogonal projection onto W. Then every column of P is an element of W. **T F**

Solution: True. The columns of P are $Pe_1, \ldots, Pe_n$, where $e_1, \ldots, e_n$ are the standard basis vectors. However, $Pe_i = \mathrm{Proj}_W(e_i) \in W$, so each column is in W.

(j) If x, y, z are vectors in $\mathbb{R}^3$ such that $x \cdot y = 0$ and $y \cdot z = 0$ then it follows that $x \cdot z = 0$. **T F**

Solution: False. Consider any nonzero vector x and any $y \perp x$, and let $x = z$. Then $x \cdot y = 0$ and $y \cdot z = y \cdot x = 0$ but $x \cdot z = x \cdot x > 0$.

2. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ denote the vector space of polynomials of degree at most 2 with coefficient-wise addition and scalar multiplication. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined by

$$T(q) = \begin{bmatrix} q(2) \\ q(-3) \end{bmatrix},$$

where $q(2)$ means the polynomial q evaluated at 2.

(a) (5 points) Find the matrix of T with respect to the bases $\mathcal{B} = \{1, t+1, t^2+t\}$ of $\mathbb{P}_2$ and $E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

Solution:

$${}_E[T]_{\mathcal{B}} = \big[[T(1)]_E [T(t+1)]_E [T(t^2+t)]_E\big] = \begin{bmatrix} 1 & 2+1 & 2^2+2 \\ 1 & -3+1 & (-3)^2-3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 1 & -2 & 6 \end{bmatrix}.$$

(b) (5 points) Find a basis for the Kernel (i.e., Null Space) of T. Explain whatever method you use.

Solution: The above matrix is row equivalent to:

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & -5 & 0 \end{bmatrix}.$$

Thus x_3 is a free variable and the Null space of ${}_E[T]_{\mathcal{B}}$ is $\{ \begin{bmatrix} -6x_3 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R}\} =$ $\mathrm{Span}\{ \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \}$. The Kernel of T is the corresponding subspace of $\mathbb{P}_2$, which is spanned by

$$-6(1) + 0(1+t) + (t^2+t) = t^2 + t - 6.$$

Thus, a basis for the Kernel of T is $\{t^2+t-6\}$.

Note that factoring this polynomial shows that it is just $(t-2)(t+3)$, which is (up to scaling by constants) the unique quadratic polynomial which vanishes at 2 and -3. This makes sense since T is just evaluating its input at these two points.

(c) (2 points) Is T 1-1? Why or why not?

Solution: No, T is not 1-1 because $\mathrm{Ker}(T) \neq \{0\}$, which means that $T(q) = 0$ has a nontrivial solution, call it q_0. Thus, for any polynomial p, we have $T(p+q_0) = T(p) + T(q_0) = T(p)$, so T is not 1-1.

(d) (3 points) Let $\mathcal{C} = \{1, t, t^2\}$. Find the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution: The change of coordinates matrix is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[[1]_{\mathcal{C}}[1+t]_{\mathcal{C}}[t^2+t]_{\mathcal{C}}\big] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

since $1 = 1\cdot 1 + 0\cdot t + 0\cdot t^2, 1+t = 1\cdot 1 + 1\cdot t + 0\cdot t^2$, and $t^2+t = 0\cdot 1 + 1\cdot t + 1\cdot t^2$.

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}.$$

(a) (2 points) Show that A is invertible.

Solution: There are many ways to see this. One is that A is in REF and has a pivot in every row. Another is that since A is upper trinagular its eigenvalues are its digonal entries, and none of them are zero, so A must be invertible. Yet another is to compute the determinant, which is the product of the diagonal entries since A is upper triangular, and which is nonzero, implying that A is invertible.

(b) (5 points) Find the eigenvalues of A^{-1}. Explain how you got your answer. (hint: there is a faster way than computing the inverse)

Solution: The trick here was to think about what an eigenvalue means. Observe that if λ is an eigenvalue of A, then $Ax = \lambda x$ for some nonzero x. Since A is invertible, we can multiply by A^{-1} on both sides to get

$$x = \lambda A^{-1} x,$$

which after rearranging is

$$A^{-1}x = (1/\lambda)x,$$

which means that $1/\lambda$ is an eigenvalue of A^{-1}. Thus, the reciprocals of the eigenvalues of A must all be eigenvalues of A^{-1}. THe eigenvalues of A, being upper triangular, are $1, 6, 10, 13, 15$, so this means that $1/1, 1/6, 1/10, 1/13, 1/15$ are eigenvalues of A^{-1}. Since A is 5×5 these are all the eigenvalues.

(c) (3 points) Is $(A^{-1})^2$ diagonalizable? Explain why or why not.

Solution: It is diagonalizable, and there are several ways to see this. One is to observe that the eigenvalues of $(A^{-1})^2$ are the squares of the eigenvalues of A^{-1}, since

$$A^{-1}x = \lambda x \quad \rightarrow A^{-1}A^{-1}x = A^{-1}\lambda x = \lambda^2 x.$$

These eigenvalues are therefore equal to $1^2, (1/6)^2, (1/10)^2, (1/13)^2, (1/15)^2$, which are distinct. This implies tht $(A^{-1})^2$ is invertible.

Another way is to observe that A^{-1} is diagonalizable since it has distinct eigenvalues, so $A^{-1} = PDP^{-1}$ for some diagonal P. But now $(A^{-1})^2 = PD^2P^{-1}$.

4. (7 points) Consider the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A. Find a diagonal matrix D an an invertible matrix P such that $AP = PD$, or explain why no such matrices exist.

This is question 5.3.15 in the book.
It is asking us to diagonalize A, since $AP=PD \iff A=PDP^{-1}$,
so the desired D will contain the eigenvalues of A
and P will contain the corresponding eigenvectors.
To find the eigenvalues, we compute the characteristic polynomial:

$$\chi_A(t) = \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ -1 & -1 & -t \end{vmatrix} = -t\Big((2-t)(-t)+1\Big) + 1\Big(-t+1\Big) - 1\Big(-1+(2-t)\Big)$$

$$= -t\left(t^2-2t+1\right) = -t(t-1)^2,$$ so the eigenvalues are $\lambda = 0, 1$.

We use row reduction (details omitted) to find the corresponding eigenspaces:

$$E_0 = \mathrm{Null}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$E_1 = \mathrm{Null}\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

So we may take

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

5. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 6 \\ 1 & 2 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

(a) (5 points) Find a least squares solution to $Ax = b$, i.e., a vector $\hat{x} \in \mathbb{R}^3$ minimizing $\|A\hat{x} - b\|$.

(b) (3 points) Using your answer to (a), or otherwise, find the orthogonal projection $\hat{b}$ of b onto the column space of A, i.e., $\hat{b} = \mathrm{Proj}_{\mathrm{Col}(A)}(b)$.