Math 54 First Midterm Exam, Prof. Srivastava
September 23, 2016, 4:10pm–5:00pm, 155 Dwinelle Hall.

Name: ____

SID: ____

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Sign here: ____

Question	Points
1	20
2	12
3	10
4	18
Total:	60

Do not turn over this page until your instructor tells you to do so.

Name and SID: ______________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   For parts (a)-(f), assume A is an arbitrary $m \times n$ matrix and R is the reduced row echelon form of A. For parts (g) and (h) assume A and B are arbitrary $n \times n$ square matrices.

   (a) A is invertible if and only if R is invertible. **T** **F**

   Solution: True. A is invertible if and only if its RREF is the identity, which is invertible.

   (b) The column space of A is equal to the column space of R.

   Solution: False. Row operations do not preserve the column space. For example, consider $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, whose RREF is $R = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   T **F**

   (c) The null space of A is equal to the null space of R. **T** **F**

   Solution: True, since the null space is the solution space of $Ax = 0$, and this is preserved by row operations.

   (d) The linear transformations $T(\mathbf{x}) = A\mathbf{x}$ and $S(\mathbf{x}) = R\mathbf{x}$ are the same. **T** **F**

   Solution: False. For several reasons, but one of them is that the column spaces are not the same, and the column space of the standard matrix is equal to the image of a linear transformation.

   (e) $\mathrm{Col}(A) = \mathbb{R}^m$ if and only if $\mathrm{Col}(R) = \mathbb{R}^m$. **T** **F**

   Solution: True. If $\mathrm{Col}(A) = \mathbb{R}^m$ then $\mathrm{rank}(A) = \mathrm{rank}(R) = m$, so the dimension of $\mathrm{Col}(R)$ must also be m, which means it must be equal to $\mathbb{R}^m$.

   Alternatively, one can observe that this condition implies that there is a pivot in every row of the RREF of A, which is R, so $Ax = b$ and $Rx = b$ are consistent for every $b \in \mathbb{R}^n$.

   (f) If $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is a basis for $\mathbb{R}^n$, then $A\mathbf{b}_1, \ldots, A\mathbf{b}_n$ must be a basis for $\mathrm{Col}(A)$. **T** **F**

   Solution: False. The reason is that $A\mathbf{b}_1, \ldots, A\mathbf{b}_n$ may not be linearly independent (though they do span $\mathrm{Col}(A)$).

(g) If A and B are invertible then $A + B$ must be invertible.

Solution: False. Consider any invertible A; then $A + (-A) = 0$ which is not invertible.

T **F**

(h) If B is invertible and every entry of A is greater than or equal to the corresponding entry of B, then A must be invertible. **T** **F**

Solution: False. Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(i) Every invertible matrix can be written as a product of elementary matrices. **T** **F**

Solution: True. This follows because every invertible matrix A can be row reduced to the identity, and since row operations correspond to elementary matrices, we have $E_k E_{k-1} \dots E_2 E_1 A = I$ for some elementary E_i. Multiplying both sides by $E_1^{-1} \dots E_k^{-1}$ gives the desired result.

(j) The set of vectors in $\mathbb{R}^4$ whose entries have a nonnegative sum: i.e.

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 \geq 0 \right\}$$

is a linear subspace of $\mathbb{R}^4$. **T** **F**

Solution: False. This set is not closed under scalar multiplication, for example, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in the set but $\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ is not.

2. For each of the following, find a pair of 2×2 matrices A and B with the specified property, or explain why no such matrices exist.

(a) (4 points) $AB \neq BA$.

Solution: There are many examples, and most pairs of matrices have this property. But consider:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and note that

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) (4 points) AB is invertible but BA is not invertible.

Solution: No such matrices exist, because if AB is invertible then BA must be invertible.

Many people simply wrote here that if AB is invertible then A and B must be invertible. This is true, but it requires a proof (note that this is logically a different statement from the converse, which says that if A and B are invertible then AB must be invertible.)

Algebraic proof: Assume AB is invertible. Then $(AB)^{-1}$ exists. Let $C = B(AB)^{-1}$, and observe that $AC = AB(AB)^{-1} = I$, so C is a right inverse of A, and by the invertible matrix theorem we must have $C = A^{-1}$. A similar argument shows that B^{-1} exists. Thus $A^{-1}B^{-1}BA = I$, so BA is invertible.

Rank based proof: Observe that if the columns of A were linearly dependent then the columns of AB would have to be linearly dependent, which is impossible since AB is invertible. Thus, the columns of A are linearly independent, and by the invertible matrix theorem A is invertible. A similar argument shows that B is invertible. Thus, $A^{-1}B^{-1}$ is an inverse of BA, as desired.

(c) (4 points) $\text{rank}(A) + \text{rank}(B) = \text{rank}(A + B)$ and $A \neq 0, B \neq 0$.

Solution: Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, which each have rank one, yet $A + B = I$, which has rank two.

3. (10 points) Consider the vectors:

$$\mathbf{v}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 2 \\ -4 \\ 7 \\ -7 \end{bmatrix}.$$

Find a vector from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ which is in the span of the remaining vectors. Express this vector as a linear combination of the remaining ones.

Solution: This question might seem hard at first, because you might wonder: which vector is it that is in the span of the others? This problem goes away when you realize that it is sufficient to find any linear dependence between the vectors, since once you have a linear combination that is zero you can move any of the vectors with a nonzero coefficient to the other side to express it as a linear combination of the rest.

To this end, we solve the system $Ax = 0$ for the matrix whose columns are $\mathbf{v}_1, \dots, \mathbf{v}_4$. Applying Row Reduction to the coefficient matrix, we have:

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -11 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

Thus x_3 is a free variable, and we have infinitely many solutions to $Ax = 0$:

$$x_1 = 11x_3 \quad x_2 = -3x_3 \quad x_4 = 0.$$

Taking $x_3 = 1$ we get the linear dependence:

$$11\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = 0,$$

so $\mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ with the linear combination

$$\mathbf{v}_3 = -11\mathbf{v}_1 + 3\mathbf{v}_2.$$

Note that this answer is not unique, since we also have

$$\mathbf{v}_1 = \frac{3}{11}\mathbf{v}_2 - \frac{1}{11}\mathbf{v}_3$$

$$\mathbf{v}_2 = \frac{11}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_3,$$

by rearranging the inequality in the two other possible ways.

Alternatively, this question can be solved by observing that the columns of the RREF have the same linear dependencies as A, so $\mathbf{v}_3 = -11\mathbf{v}_1 + 3\mathbf{v}_2$.

4. Consider the linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates a vector counterclockwise by $\pi/2$ radians.

(a) (5 points) Let $T = T_2 \circ T_1$, $T : \mathbb{R}^3 \to \mathbb{R}^2$ denote the composition of T_1 and T_2. Find the standard matrix of T. Call this matrix A.

Solution: To find the standard matrix, we compute

$$T_2(T_1(\begin{bmatrix}1\\0\\0\end{bmatrix})) = T_2(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix},$$

$$T_2(T_1(\begin{bmatrix}0\\1\\0\end{bmatrix})) = T_2(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}-1\\1\end{bmatrix},$$

$$T_2(T_1(\begin{bmatrix}0\\0\\1\end{bmatrix})) = T_2(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}-1\\0\end{bmatrix},$$

so the standard matrix is

$$A = \begin{bmatrix}0 & -1 & -1\\1 & 1 & 0\end{bmatrix}.$$

(b) (5 points) Find bases for Col(A) and Null(A).

Solution: After swapping the rows, the RREF of A is:

$$A = \begin{bmatrix}1 & 1 & 0\\0 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\end{bmatrix}.$$

The first two columns are pivot columns, and a basis for Col(A) is given by the corresponding columns of A, namely:

$$\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}.$$

Note that I did not specifically ask for a basis coming from the columns of A, so $\mathbf{e}_1, \mathbf{e}_2$ is also an acceptable answer here. But it is important to note that conceptually the columns of the RREF may have nothing to do with Col(A) *– it just works out in this case because* $\mathrm{Col}(A) = \mathbb{R}^2$.

To find a basis for the null space, we consider that the general solution to $Ax = 0$ is given by

$$\mathrm{Null}(A) = \left\{\begin{bmatrix}x_3\\-x_3\\x_3\end{bmatrix} : x_3 \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\},$$

so

$$\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

is a basis for Null(A).

(c) (2 points) Is T onto? Explain why in terms of your answers to part (b).

Solution: Yes, T is onto because the image of T is equal to column space of its standard matrix A, which is $\mathbb{R}^2$ from the previous part since the basis contains two vectors.

Alternatively, you could just say that for every b in the codomain there is an x such that $T(x) = Ax = b$, because $Ax = b$ is always consistent since the RREF of A has a pivot in every row.

(d) (6 points) Find two **distinct** vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ such that $T(\mathbf{v}_1) \neq 0$, $T(\mathbf{v}_2) \neq 0$, and $T(\mathbf{v}_1) = T(\mathbf{v}_2)$.

Solution: *There are many ways to do this, and below is just one. Another is simply to choose a vector b in the image of T, solve $Ax = b$, and observe that it has multiple solutions.*

We first find one vector $\mathbf{v}_1$ such that $T(\mathbf{v}_1) \neq 0$. Let's just take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ since for this vector we have already computed $T(\mathbf{v}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. To find another vector, we note that for any vector $\mathbf{w} \in \text{Null}(A)$, we have

$$T(\mathbf{v}_1 + \mathbf{w}) = T(\mathbf{v}_1) + T(\mathbf{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + A\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

since $A\mathbf{w} = 0$. Choosing

$$\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

from part (b) we have

$$T(\mathbf{v}_2) = T(v_1) + T(\mathbf{w}) = T(\mathbf{v}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{w} \neq \mathbf{v}_1$, as desired.