1. (20 points) Circle always true (T) or sometimes false (F) for each of the following. There is no need to provide an explanation. Two points each.

For parts (a)-(f), assume $A$ is an arbitrary $m \times n$ matrix and $R$ is the reduced row echelon form of $A$. For parts (g) and (h) assume $A$ and $B$ are arbitrary $n \times n$ square matrices.

(a) $A$ is invertible if and only if $R$ is invertible. \hspace{1cm} T \hspace{0.5cm} F

(b) The column space of $A$ is equal to the column space of $R$. \hspace{1cm} T \hspace{0.5cm} F

(c) The null space of $A$ is equal to the null space of $R$. \hspace{1cm} T \hspace{0.5cm} F

(d) The linear transformations $T(x) = Ax$ and $S(x) = Rx$ are the same. \hspace{1cm} T \hspace{0.5cm} F

(e) $\text{Col}(A) = \mathbb{R}^m$ if and only if $\text{Col}(R) = \mathbb{R}^m$. \hspace{1cm} T \hspace{0.5cm} F

(f) If $b_1, \ldots, b_n$ is a basis for $\mathbb{R}^n$, then $Ab_1, \ldots, Ab_n$ must be a basis for $\text{Col}(A)$. \hspace{1cm} T \hspace{0.5cm} F

(g) If $A$ and $B$ are invertible then $A + B$ must be invertible. \hspace{1cm} T \hspace{0.5cm} F

(h) If $B$ is invertible and every entry of $A$ is greater than or equal to the corresponding entry of $B$, then $A$ must be invertible. \hspace{1cm} T \hspace{0.5cm} F

(i) Every invertible matrix can be written as a product of elementary matrices. \hspace{1cm} T \hspace{0.5cm} F

(j) The set of vectors in $\mathbb{R}^4$ whose entries have a nonnegative sum: i.e.

\[
\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 \geq 0 \right\}
\]

is a linear subspace of $\mathbb{R}^4$. \hspace{1cm} T \hspace{0.5cm} F
2. For each of the following, find a pair of $2 \times 2$ matrices $A$ and $B$ with the specified property, or explain why no such matrices exist.

(a) (4 points) $AB \neq BA$.

(b) (4 points) $AB$ is invertible but $BA$ is not invertible.

(c) (4 points) $\text{rank}(A) + \text{rank}(B) = \text{rank}(A + B)$ and $A \neq 0, B \neq 0$. 
3. (10 points) Consider the vectors:

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 9 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ -4 \\ -7 \end{bmatrix}.
\end{align*}
\]

Find a vector from \( v_1, v_2, v_3, v_4 \) which is in the span of the remaining vectors. Express this vector as a linear combination of the remaining ones.
4. Consider the linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates a vector counterclockwise by $\pi/2$ radians.

(a) (5 points) Let $T = T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ denote the composition of $T_1$ and $T_2$. Find the standard matrix of $T$. Call this matrix $A$.

(b) (5 points) Find bases for $\text{Col}(A)$ and $\text{Null}(A)$.
(c) (2 points) Is $T$ onto? Explain why in terms of your answers to part (b).

(d) (6 points) Find two distinct vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ such that $T(\mathbf{v}_1) \neq 0$, $T(\mathbf{v}_2) \neq 0$, and $T(\mathbf{v}_1) = T(\mathbf{v}_2)$.