MATH 54 Homework 9 Solutions

10/19

6.1

6.1.2

$w \cdot w = 35$
$x \cdot w = 5$
$\frac{x \cdot w}{w \cdot w} = \frac{1}{7}$

6.1.6

$x \cdot x = 49$
$\frac{x \cdot w}{x \cdot x} x = \begin{bmatrix} 30/49 \\ -(10/49) \\ 15/49 \end{bmatrix}$

6.1.7

$\|w\| = \sqrt{35}$

6.1.10

$w = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \|w\| = \sqrt{61}$

$\frac{1}{\sqrt{61}} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

6.1.14

$\|u - z\| = 2\sqrt{17}$

6.1.17

They are orthogonal since $u \cdot v = -12 + 2 + 10 = 0$

6.1.19

a. True
b. True
c. True We have $(u-v) \cdot (u-v) = (u+v) \cdot (u+v)$. Expanding this we get $u \cdot u + v \cdot v - 2u \cdot v = u \cdot u + v \cdot v + 2u \cdot v$. Simplyfiny we get $4u \cdot v = 0$, so they are orthogonal.
d. False Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $Nul(A) = \text{Span} e_1$ and $Col(A) = \text{Span} e_1$.
e. True. If $x \cdot v_j = 0$, then given any $w \in W$ we have $w = \sum c_i v_i$ so $x \cdot w = \sum_i c_i (x \cdot v_i) = 0$.

6.1.24

$$\|u+v\|^2 + \|u-v\|^2 = (u+v) \cdot (u+v) + (u-v) \cdot (u-v) = 2u \cdot u + 2v \cdot v + 2u \cdot v - 2u \cdot v = 2\|u\|^2 + 2\|v\|^2$$

6.1.27

We have $y \cdot u = y \cdot v = 0$. Then $y \cdot (u + v) = y \cdot u + y \cdot v = 0$.

6.1.29

We have $x \cdot v_j = 0$. Then given any $w \in W$ we have $w = \sum c_j v_j$.

$$x \cdot w = x \cdot \left(\sum c_j v_j\right) = \sum x \cdot c_j v_j = \sum c_j (x \cdot v_j) = 0$$

6.2

6.2.5

The dot product of all three pairs are 0, so the vectors are orthogonal.

6.2.9

$$\begin{aligned} \text{proj}_{u_1} x + \text{proj}_{u_2} x + \text{proj}_{u_3} x &= x \\ \frac{5}{2} u_1 + \frac{-3}{2} u_2 + 2u_3 &= x \end{aligned}$$

6.2.23abc

a. True Consider $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
b. True For $x \in \text{Span}\{u_1, \ldots, u_k\}$ we have $x = \sum_i \text{proj}_{u_i} x$
c. False If $u \cdot v = 0$ then $(cu) \cdot v = 0$.

6.2.32

v_1, v_2 are an orthogonal set if and only if $v_1 \cdot v_2 = 0$ and $v_1, v_2 \neq 0$. Then sincce $c_1 v_1, c_2 v_2 \neq 0$, we only need to check that $(c_1 v_1) \cdot (c_2 v_2) = c_1 c_2 (v_1 \cdot v_2) = 0$.

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6.1

6.1.30

We have $W^\perp = \{x \in \mathbb{R}^n \mid x \cdot w = 0, w \in W\}$. First we show $0 \in W^\perp$. To see this we require $0 \cdot w = 0$ for all $w \in W$.

To show $W^\perp$ is closed under addition, let $z_1, z_2 \in W^\perp$ be given. Then $z_1 \cdot w = z_2 \cdot w = 0$ for all $w \in W$. We see $(z_1 + z_2) \cdot w = z_1 \cdot w + z_2 \cdot w = 0$ for all $w \in W$, so $z_1 + z_2 \in W^\perp$.

To show $W^\perp$ is closed under scalar multiplication, let $z \in W^\perp$ be given. Then $z \cdot w = 0$ for all $w \in W$. Let $c \in \mathbb{R}$, so $(cz) \cdot w = c(z \cdot w) = 0$ for all $w \in W$ which implies $cz \in W^\perp$.

6.3

6.3.1

We know that $\text{proj}_{u_4}(x) = \frac{u_4 \cdot x}{u_4 \cdot u_4} u_4 \in \text{Span}\{u_4\}$. Then

$$\text{proj}_{u_4}(x) = \frac{50 + 24 - 2}{25 + 9 + 1 + 1} u_4 = \frac{72}{36} u_4 = 2u_4.$$

Also since $x = \sum_i \text{proj}_{u_i}(x)$ we have $x - 2u_4 \in \text{Span}\{u_1, u_2, u_3\}$.

$$\begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

6.3.4

To see $\{u_1, u_2\}$ is orthogonal we verify $u_1 \cdot u_2 = -12 + 12 + 0 = 0$. Then

$$\text{proj}_{\text{Span}\{u_1,u_2\}} = \text{proj}_{u_1}(y) + \text{proj}_{u_2}(y) = \frac{18+12}{25}u_1 + \frac{-24+9}{25}u_1 = \frac{1}{5}\begin{bmatrix} 18+12 \\ 24-9 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

6.3.8

Note that u_1, u_2 are orthogonal, so

$$\text{proj}_W y = \text{proj}_{u_1}(y) + \text{proj}_{u_2} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{-1}{2} \\ \frac{3}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix}$$

We know that $y - \text{proj}_W y \in W^\perp$ so we get the final answer

$$y = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{-5}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}$$

6.3.12

Note that v_1 is orthogonal to v_2. Then the desired point is

$$\text{proj}_{\text{Span}\{v_1,v_2\}} y = \text{proj}_{v_1} v_1 + \text{proj}_{v_2} v_2 = 3v_1 + v_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

6.3.14

Note that v_1 is orthogonal to v_2. Then the desired point is

$$\text{proj}_{\text{Span}\{v_1,v_2\}} z = \text{proj}_{v_1} v_1 + \text{proj}_{v_2} v_2 = \frac{1}{2}v_1 + 0v_2 = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

6.3.19

Let $W = \text{Span}\{u_1, u_2\}$. We have $u_3 - \text{proj}_W u_3 \in W^\perp$. Since $u_3 \notin W$, we have $u_3 - \text{proj}_W u_3 \neq 0$.

$$u_3 - \text{proj}_W u_3 = u_3 - \text{proj}_{u_1} u_3 - \text{proj}_{u_2} u_3 = \begin{bmatrix} -\frac{3}{2} \\ -\frac{7}{2} \\ 0 \end{bmatrix}$$

6.3.22

a. True If $v \in W, W^{\perp}$ we have $v \cdot v = 0$, so $v = 0$.
b. True The terms are projections onto the one dimensional subspaces of W spanned by the orthogonal basis elements.
c. True The orthogonal decomposition is unique.
d. False The best approximation is $\text{proj}_W y$. $y - \text{proj}_w y \in W^p erp$.
e. False Consider $U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $UU^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

6.3.24

Since $\{w_1, \ldots, w_p\}$ is orthogonal and $\{v_1, \ldots, v_q\}$ is orthogonal we only need to check that $w_i \cdot v_j = 0$. This follows since $v_j \in W^{\perp}$ and $w_i \in W$.

To see that $\{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ spans $\mathbb{R}^n$ let $v \in \mathbb{R}^n$ be given. By the decomposition theorem we have $v = \text{proj}_W v + z$ with $\text{proj}_W v \in W, z \in W^{\perp}$. Hence we can write proj_W as a linear combination of $w_1, \ldots, w_p$ and z as a linear combination of $v_1, \ldots, v_q$. Combining these we can write v as a linear combination of $w_1, \ldots, w_p, v_1, \ldots, v_q$.

Thus we see the set $\{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ spans $\mathbb{R}^n$ and since its orthogonal we know its linearly independent, so its a basis of $\mathbb{R}^n$. Hence it has n elements so $p + q = n$.

10/24

6.4

6.4.2

$v_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

$v_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$

Normalize the vectors:

$\tilde{v_1} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

$\tilde{v_2} = \frac{1}{\sqrt{105}} \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$

6.4.6

$v_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

$v_2 = \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$

Normalize the vectors:

$v_1 = \frac{1}{\sqrt{15}} \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{61}} \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$

6.4.10

Apply Gram-Schmidt to the column vectors to get the orthogonal set:

$$\frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

6.4.17ab

a. False If $c = 0$ then it will make the linearly independent set linearly dependent.
b. True

6.4.22

We have $T(0) = \text{proj}_W 0 = 0$ since $0 \in W$.

Given $x, y \in \mathbb{R}^n$, we have $x = \text{proj}_W x + w$, $y = \text{proj}_W y + z$ where $w, z \in W^\perp$. By the decomposition theorem these are unique. Hence we have $x + y = (\text{proj}_W x + \text{proj}_W y) + (z + w)$. Since $W, W^\perp$ are subspaces, we have $(\text{proj}_W x + \text{proj}_W y) \in W$ and $z + w \in W^\perp$. This implies by the decomposition theorem that $\text{proj}_W x + \text{proj}_W y = \text{proj}_W(x + y)$, so $T(x + y) = T(x) + T(y)$.

Similarly, given $x \in \mathbb{R}^n$ we have $x = \text{proj}_W x + z$ where $z \in W^\perp$, so $cx = c\text{proj}_W x + cz$. Since $c\text{proj}_W x \in W$ and $cz \in W^\perp$, by decomposition theorem we have $c\text{proj}_W x = \text{proj}_W(cx)$, so $T(cx) = cT(x)$.

6.5

6.5.3

We calculate $A^T A = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$, $A^T b = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$. We solve $A^T A x = A^T b$ to find

$$x = \begin{bmatrix} \frac{4}{3} \\ \frac{-1}{3} \end{bmatrix}$$

6.5.5

We calculate $A^TA = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 14 \\ 2 \\ 10 \end{bmatrix}$. Then solutions to $A^TAx = A^Tb$ are a particular solution plus the nullspace. We see that this means

$$x = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

6.5.7

We compute $\|Ax - b\| = 2\sqrt{5}$

6.5.9

$A^TA = \begin{bmatrix} 14 & 0 \\ 0 & 42 \end{bmatrix}$
$A^Tb = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$
Then we solve $A^TAx = A^Tb$ to find

$$x = \begin{bmatrix} \frac{2}{7} \\ \frac{-1}{7} \end{bmatrix}$$

The orthogonal projection of b onto $Col(A)$ is therefore $Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

6.5.17

a. True
b. True
c. False We want this norm to minimized, not maximized
d. True
e. True We have the rank of A is the same as the rank of A^TA, so when the columns of A are linearly independent we have the rank of A^TA is maximal, and hence A^TA is invertible, so $A^TAx = A^Tb$ has a unique solution.

6.5.19

If $Ax = 0$ then we have $A^TAx = A^T0 = 0$.
If $A^TAx = 0$ then we have $x^TA^TAx = x^T0 = 0$. We note $0 = x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2$ which implies $Ax = 0$.

6.5.22

Since the number of columns of A and A^TA Are the same, by 6.5.19 we have concluded the nullity of A and A^TA are equal, so by rank nullity the rank of A and A^TA are equal.