5.3.5 The eigenvalues are $3, 2, 3$ (i.e., the diagonal entries of D) with corresponding eigenvalues $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

5.3.8 The matrix is upper triangular so the eigenvalues are simply the diagonal entries, namely $3, 3$. The corresponding eigenspace is given by

$$Null(A - 3I) = Null(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}),$$

which is the span of e_1. Since there is no basis of eigenvectors, this matrix is not diagonalizable.

5.3.13 The eigenspaces are

$$E_1 = Null \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} = Span\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

and

$$E_5 = Null \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} = Span\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \}.$$

Since there are three linearly independent eigenvectors (since eigenvectors from distinct eigenspaces are always linearly independent), the matrix can be diagonalized by taking D with diagonal entries equal to $1, 1, 5$ and P with columns as above, corresponding to the eigenvalues.

5.3.14 The eigenspaces are

$$E_2 = Null \begin{bmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = Span\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$E_3 = Null \begin{bmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = Span\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

Since we have a basis of three eigenvectors, the matrix is diagonalizable D containing the eigenvalues as diagonal entries and P containing the above eigenvectors as the corresponding columns.

5.3.21 (a) False. D must be a *diagonal* matrix. (b) True. See Theorem 5. (c) False. Diagonalizability is about having enough linearly independent *eigenvectors*, not eigenvalues. (d) False. A is invertible iff A does not have zero as an eigenvalue, which has nothing to do with diagonalizability.

5.3.27 Since A is diagonalizable, we can write $A = PDP^{-1}$ for some diagonal D whose entries are the eigenvalues of A. Since A is invertible none of the eigenvalues are zero. Thus, D has exactly n pivots so it is invertible, i.e., D^{-1} exists (alternatively, you can just consider the diagonal matrix with entries $1/\lambda_i$ if D has entries λ_i on the diagonal). Now observe that

$$(PD^{-1}P^{-1})A = PD^{-1}P^{-1}PDP^{-1} = PD^{-1}DP^{-1} = PP^{-1} = I,$$

so A has a left inverse. By the invertible matrix theorem (or by direct calculation) this implies that A also has a right inverse, and is invertible.

5.3.31 Consider the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. This matrix is already in REF and has a pivot in every row, so it is invertible. Since it is upper triangular its eigenvalues are equal to its diagonal entries, so its only eigenvalue is 2. However the corresponding eigenspace is

$$E_2 = Null \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = Span\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \},$$

so the matrix does not have a basis of eigenvectors and is not diagonalizatable.

5.3.32 Any diagonalizable matrix A with zero as an eigenvalue will do, since this implies A is not invertible. Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Since this matrix is upper triangular its eigenvalues are just $1, 0$, and since they are distinct the matrix must be diagonalizable. (One can also observe that the matrix is in REF and does not have a pivot in every row.)

5.4.11 The $B-$matrix of $x \mapsto Ax$ is given by

$$[A]_B = P_{B \leftarrow E} A P_{E \leftarrow B},$$

where A is the standard matrix. We have

$$P_{E \leftarrow B} = P_B = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

and

$$P_{B \leftarrow E} = P_B^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix},$$

so multiplying we obtain

$$[A]_B = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}.$$

5.4.14 We compute the characteristic polynomial

$$\chi_A(t) = \det \begin{bmatrix} 2-t & 3 \\ 3 & 2-t \end{bmatrix} = (2-t)^2 - 9 = t^2 - 4t - 5 = (t-5)(t+1).$$

Thus the eigenvalues are $5, -1$, and since they are distinct the matrix must be diagonalizable. The corresponding eigenvectors are given by solving:

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v_2 = 0 \quad \Rightarrow \quad v_2 = c \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So $[T]_B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ in the basis $B = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

5.4.17 See answer key.

5.4.23 Suppose $Ax = \lambda x$. We calculate

$$B(P^{-1}x) = P^{-1}APP^{-1}x = P^{-1}Ax = P^{-1}(\lambda x) = \lambda P^{-1}x.$$

Moreover, $P^{-1}x \neq 0$ since P^{-1} is invertible and $x \neq 0$. Thus $P^{-1}x$ is an eigenvector of B with eigenvalue λ, as desired.

5.3.2 $A^4 = PD^4P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}.$

5.3.3 See answer key.

5.3.4 Noting that $A^k = PD^kP^{-1}$ and that $D^k = \begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix}$, we obtain:

$$A^k = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3(-3)^k + 4(-2)^k & 6(-3)^k - 6(-2)^k \\ -2(-3)^k + 2(-2)^k & 4(-3)^k - 3(-2)^k \end{bmatrix}.$$

5.3.5 See answer key.

5.3.22 (a) False. You need n *linearly independent* eigenvectors. (b) False. Consider the identity. (c) True. Let $v_1, \ldots, v_n$ be the columns of P. Then the columns of AP are $Av_1, \ldots Av_n$ and the columns of PD are $d_i v_i$ where d_i is the i^{th} diagonal entry of D. Comparing columns we find that $Av_i = d_i v_i$, which means that the v_i are eigenvectors and the d_i eigenvalues, whenever $v_i \neq 0$. (d) False. See 5.3.31.

5.3.24 No, it is not, since it has at most two linearly independent eigenvectors (one from each eigenspace).

5.3.25 No it is not possible, i.e., A *must* be diagonalizable. Let the three eigenvalues be a, b, c with eigenspaces E_a, E_b, E_c. Assume E_a has dimension 1 and E_b has dimension two. Since c is an eigenvalue E_c must have dimension at least one. Thus the sum of the dimensions of the eigenspaces is 4, so by Theorem 7 the matrix must be diagonalizable.

5.3.29 See answer key. Basically, the idea is that you can switch the order of the eigenvalues and eigenvectors and get matrices P and D with correspondingly switched columns and rows. Another way to get a different P_1 would be to use some other scaling of the eigenvectors (e.g., $P_1 = \begin{bmatrix} 10 & 3 \\ -10 & -6 \end{bmatrix}$)

5Supp.2 Assume $ABx = \lambda x$. Observe that $BA(Bx) = BABx = B(\lambda x) = \lambda(Bx)$, so since $Bx \neq 0$ we conclude that Bx is an eigenvector of BA.

5.Supp9 Observe that if x is an eigenvector of A with eigenvalue λ, then $Ax = \lambda x$, which is equivalent to $x - Ax = x - \lambda x$ which is the same as $(I - A)x = (1 - \lambda)x$. Thus the eigenvalues of $I - A$ are exactly $1 - \lambda_i$ where λ_i are the eigenvalues of A. Since $|\lambda_i| < 1$ for all the eigenvalues, 1 is not an eigenvalue of A, and consequently $1 - 1 = 0$ is not an eigenvalue of $I - A$, which means that $I - A$ is invertible.

5.Supp10 This question assumes quite a bit of familiarity with limits, which is why it was optional (and such questions will not be on the exam). The statement $A^k \to 0$ means that

$$\lim_{k\to\infty} (A^k)(i, j) == 0,$$

for every $i, j \leq n$, i.e., every entry of the sequence $A, A^2, A^3, \ldots$ tends to zero. Fix i, j and observe that the i, j entry of A^k is equal to $e_i^T A^k e_j$. Since A is diagonalizable, we have

$$e_i^T A^k e_j = e_i^T P D^k P^{-1} e_j = v^T D^k w,$$

letting $v = P^T e_i$ and $w = P^{-1} e_j$. Suppose $\lambda_1, \ldots, \lambda_n$ are the diagonal entries of D (i.e., the eigenvalues) and recall that $|\lambda_i| < 1$. We now have

$$|v^T D^k w| = |\lambda_1^k v_1 w_1 + \lambda_2^k v_2 w_2 + \ldots + \lambda_n^k v_n w_n| \leq |\lambda_1|^k |v_1||w_1| + |\lambda_2|^k |v_2||w_2| + \ldots + |\lambda_n|^k |v_n||w_n|.$$

Since $|\lambda_i|^k \to 0$ as $k \to \infty$, the sum on the right hand side must tend to zero, so each entry of A^k must also tend to zero, as desired.

5.5.1 See answer key.

5.5.4 To find the eigenvalues we first find the characteristic polynomial

$$\chi_A(t) = \det \begin{bmatrix} 1-t & -2 \\ 1 & 3-t \end{bmatrix} = (1-t)(3-t) + 2 = t^2 - 4t + 5,$$

which has roots $\frac{4\pm\sqrt{16-20}}{2} = 2 \pm i$. We find the corrresponding eigenvectors by row reduction; for $\lambda = i + 1$ we have

$$\begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \Rightarrow \begin{bmatrix} -1-i & -2 \\ 0 & 1-i-\frac{-2}{-1-i} \end{bmatrix} \Rightarrow \begin{bmatrix} -1-i & -2 \\ 0 & 0 \end{bmatrix},$$

since by complex arithmetic we have $(1-i)(-1-i) = -(1-i^2) = -2$, which implies that $\frac{-2}{-1-i} = 1-i$, so the bottom right entry is zero. Thus the second variable is free and the nullspace is given by

$$E_{1+i} = \{\begin{bmatrix} \frac{2}{-1-i}x_2 \\ x_2 \end{bmatrix} : x_2 \in \mathbb{C}\} = Span\{\begin{bmatrix} \frac{2}{-1-i} \\ 1 \end{bmatrix}\} = Span\{\begin{bmatrix} -1+i \\ 1 \end{bmatrix}\}.$$

Thus $Av_1 = (1+i)v_1$ where $v_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$.

We could find the eigenvector corresponding to the other eigenvalue $1-i$ by row-reduction, but a more conceptual way is to observe that

$$A\overline{v_1} = \overline{Av_1} = \overline{(1+i)v_1} = (1-i)\overline{v_1},$$

so the other eigenvector must simply be $v_2 = \overline{v_1} = \begin{bmatrix} \overline{-1+i} \\ \overline{1} \end{bmatrix} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$.

5.5.8 Computing $r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9+27} = 6$, we have

$$\begin{bmatrix} 3 & 3\sqrt{3} \\ -3\sqrt{3} & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} = \frac{1}{6}\left[\cos -\pi/3 \quad -\sin(-\pi/3\right] \sin(-\pi/3)\cos(\pi/3),$$

so the given matrix corresponds to a scale factor of $r = 6$ and a rotation by $\phi = -\pi/3$.

5.5.12 Just as above, we calculate $r = \sqrt{3^2 + \sqrt{3}^2} = 2\sqrt{3}$ and $\cos\phi = 3/(2\sqrt{3}) = \sqrt{3}/2$, $\sin\phi = \sqrt{3}/(2\sqrt{3}) = 1/2$, yielding $\phi = \pi/6$.

5.5.13 See answer key.

5.5.22 Assume $Ax = \lambda x$ and let $\mu \in \mathbb{C}$. Observe that

$$A(\mu x) = \mu Ax = \mu\lambda x = \lambda(\mu x),$$

since matrix vector multiplication has the same properties over $\mathbb{C}$ as over $\mathbb{R}$, and complex number multiplication is commutative. Since $\mu \neq 0$ $\mu x \neq 0$ and we conclude that μx is an eigenvalue of A with eigenvalue λ.

5.5.25 See answer key.