Homework 7 Answers
Math 54 - Fall 2016

Section 4.7

4.7.3 With $\mathcal{U} = \{u_1, u_2\}, \mathcal{W} = \{w_1, w_2\}$. $P[x]_{\mathcal{W}} = [x]_{\mathcal{U}}$.

Think of if the action of P on the basis $\mathcal{U}$. P on $[u_1]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ will return the first column of P - which is $[u_1]_{\mathcal{W}}$. So (ii.) should be satisfied, but not (i.).

4.7.7 $P_B = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}, P_C = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}$. $P_{C\leftarrow B} = P_C^{-1}P_B$. and $P_{B\leftarrow C} = P_B^{-1}P_C$. We can either compute these by computing the inverses, and then multiplying or by row-reducing the matrices

$$\left[\ P_C \mid P_B\ \right] \sim \left[\ I \mid P_{C\leftarrow B}\ \right]$$

and likewise for $P_{B\leftarrow C}$. In any case, $P_C^{-1} = \begin{bmatrix} -1/4 & -1/4 \\ -5/8 & -1/8 \end{bmatrix}, P_B^{-1} = \begin{bmatrix} -1/8 & 3/8 \\ -5/8 & 7/8 \end{bmatrix}$ so

$$P_{C\leftarrow B} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}, P_{B\leftarrow C} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}.$$

4.7.8 $P_B = \begin{bmatrix} -1 & 1 \\ 8 & -7 \end{bmatrix}, P_C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. $P_{C\leftarrow B} = P_C^{-1}P_B$. and $P_{B\leftarrow C} = P_B^{-1}P_C$. We can either compute these by computing the inverses, and then multiplying or by row-reducing the matrices

$$\left[\ P_C \mid P_B\ \right] \sim \left[\ I \mid P_{C\leftarrow B}\ \right]$$

and likewise for $P_{B\leftarrow C}$. In any case, $P_C^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}, P_B^{-1} = \begin{bmatrix} 7 & 1 \\ 8 & 1 \end{bmatrix}$ so

$$P_{C\leftarrow B} = \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}, P_{B\leftarrow C} = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}.$$

4.7.13 The change of basis matrix from B to standard coordinates is the matrix with columns of B represented in standard coordinates. So $P_{S\leftarrow B} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$. To find the B-coordinates of $-1+2t$ we are looking for the solution to $Pv = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Row reducing this linear system, we have $v = \begin{bmatrix} 41 \\ -14 \\ 5 \end{bmatrix}$.

4.7.14 As in the previous problem, the change of basis matrix to standard coordinates has columns the vectors of B in standard coordinates. So $P_{S\leftarrow B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$. To find B-coordinates of t^2 we are looking for the v so that $Pv = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Row-reducing, we get $v = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

4.7.20.a $P_{D\leftarrow B} = P_{D\leftarrow C}P_{C\leftarrow B}$. The matrix on the left takes in vectors in B coordinates, and outputs their D coordinates. The product on the right takes in vectors with B coordinates, converts them to C coordinates, then converts them to D coordinates. This is the same as just converting them to D coordinates right away. Alternatively, $P_{D\leftarrow C}P_{C\leftarrow B} = P_D^{-1}P_CP_C^{-1}P_B = P_D^{-1}P_B = P_{D\leftarrow B}$.

Section 4.5

4.5.9 The subspace is the set of $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$. This is clearly spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. These are linearly independent, so the dimension is 2.

4.5.11 This is just the dimension of column space of

$$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix}.$$

This row-reduces to

$$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of pivot columns is 3 so the dimension is 3.

4.5.25 $B = \{1, 1-t, 2-4t+t^2\}$. $p = 5+5t-2t^2$. To find the B coordinates, we solve the linear system $a_1(1) + a_2(1-t) + a_3(2-4t+t^2) = p$. So we row-reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -4 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

So the coordinates are $\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$.

4.5.26 Pick a basis for H, say B. B is a linearly independent set. Suppose v is in V, but not in H. Then $B \cup \{v\}$ is linearly independent, since v is not in the span of B - H. But then we would have $n+1$ linearly independent vectors in an n-dimensional vector space. This is silly. So $H = V$.

4.5.27 The space of polynomials is infinite-dimensional space, because any basis would have to have $1, x, x^2, x^3, ...,$ as linear combinations of its elements. As these elements are linearly inepenent, there have to be at least as many basis elements as these - so infinitely many.

4.5.30 a.) No. $\{1, 1\}$ is a linearly dependent set in $\mathbb{P}^2$, but clearly this has dimension greater than 2. b.)Yes. If every set of p elements cannot span V, then any basis of V must have more than p elements. So $\dim V > p$. c.) No. In $\mathbb{P}^2$, $\{1, 1\}$ is linearly dependent.

4.5.31x Pick a basis for $T(H)$, say $w_1, ..., w_k$. Then there exist $v_1, ..., v_k$ in H so that $T(v_i) = w_i$. T maps linearly dependent sets to linearly dependent sets. So $v_1, .., v_k$ must be linearly independent, since w_i are. A basis is a maximally linearly independent set, so $\dim H \geq k = \dim T(H)$.

Section 3.1

3.1.4

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 1(2-4) - 3(4-3) + 5(8-3) = 20$$

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = -3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -3(4-3) + 1(2-15) - 4(1-10) = -3-13+36 = 20$$

3.1.22

$$\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = ad + kcd - bc - kdc = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Adding a multiple of a row to another row does not change the determinant.

3.1.25

$$\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & k & 1\end{vmatrix} = 1\begin{vmatrix}1 & 0\\ k & 1\end{vmatrix} - 0\begin{vmatrix}0 & 0\\ k & 1\end{vmatrix} + 0\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix} = 1(1-0k) = 1.$$

3.1.38 $\begin{vmatrix}ka & kb\\ kc & kd\end{vmatrix} = k^2ad - k^2ab = k^2\begin{vmatrix}a & b\\ c & d\end{vmatrix}.$

3.1.43x No this is not true. Note that

$$\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} + \begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix} = 0 + 0 \neq 1 = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}.$$

Section 3.2

3.2.6

$$\begin{vmatrix}1 & 5 & -3\\ 3 & -3 & 3\\ 2 & 13 & -7\end{vmatrix} = \begin{vmatrix}1 & 5 & -3\\ 0 & -18 & 12\\ 0 & 3 & -1\end{vmatrix} = \begin{vmatrix}1 & 5 & -3\\ 0 & 0 & 6\\ 0 & 3 & -1\end{vmatrix} = 6\begin{vmatrix}1 & 5 & 0\\ 0 & 0 & 1\\ 0 & 3 & 0\end{vmatrix} = 6(3)\begin{vmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{vmatrix} = -6(3)\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = -18$$

3.2.7

$$\begin{vmatrix}1 & 3 & 0 & 2\\ -2 & -5 & 7 & 4\\ 3 & 5 & 2 & 1\\ 1 & -1 & 2 & -3\end{vmatrix} = \begin{vmatrix}1 & 3 & 0 & 2\\ 0 & 1 & 7 & 8\\ 0 & -4 & 2 & -5\\ 0 & -4 & 2 & -5\end{vmatrix} = \begin{vmatrix}1 & 3 & 0 & 2\\ 0 & 1 & 7 & 8\\ 0 & 0 & 30 & 27\\ 0 & 0 & 0 & 0\end{vmatrix} = 0,$$

since the determinant of an upper triangular matrix is the product of the diagonal elements.

3.2.16

$$\begin{vmatrix}a & b & c\\ 3d & 3e & 3f\\ g & h & i\end{vmatrix} = 3\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 3 * 7 = 21.$$

3.2.22

$$\begin{vmatrix}5 & 0 & -1\\ 1 & -3 & -2\\ 0 & 5 & 3\end{vmatrix} = 5\begin{vmatrix}-3 & -2\\ 5 & 3\end{vmatrix} - \begin{vmatrix}1 & -3\\ 0 & 5\end{vmatrix} = 5(-9+10) - (5-0) = 0.$$

So the matrix is not invertible.

3.2.27 a.) Row replacement may affect a determinant. Scaling the row scales the determinant. b.) This is true. Only scaling and swapping affect the determinant. Swapping is taken care of by the -1 product, and the scaling is just the product of the pivots. c.) If the columns of A are linearly dependent, the matrix is not invertible, so the determinant is 0. d.) This is false. The example in 3.1.43x is a counterexample.

3.2.29

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & = \end{vmatrix} - 2 * \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2.$$

The determinant of a product is the product of the determinants, so $\det B^5 = -32$.

3.2.32 $\det(rA) = r^n \det A$. Thm 3 on all n rows.

3.2.33x $\det(AB) = \det A \det B = \det B \det A = \det(BA)$. Thm 6

3.2.34 $\det(PAP^{-1}) = \det(PA) \det P^{-1} = \det P^{-1} \det(PA) = \det(P^{-1}P) \det A = 1 \det A$. Thm 6

Section 5.1

5.1.2 $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$. -3 not because $\begin{bmatrix} 2 & 4 \\ 6 & 6 \end{bmatrix}$ is rank 2, so is invertible.

5.1.7

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 4.

5.1.16

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 4 & -2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the eigenvectors for $\lambda = 4$ are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix}$$

Taking $a = 1$, we get a basis for this eigenspace.

5.1.20 0 is an eigenvalue with eigenvectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. This matrix has rank 1 so the eigenspace for 0 has dimension 2. Clearly these vectors are linearly independent.

5.1.21 a.) True, this is the definition of an eigenvalue. b.) True. The 0 eigenspace is just the null space. If it is not-trivial, then it is not invertible. c.)True. If $(A - cI)v = 0$, for some non-zero v, then $Av = cv$, so c is an eigenvalue. d.) True. If v is an eigenvector, $Av = \lambda v$, so we can check for linear dependence quickly. e.) False. To find eigenvalues, check when $A - \lambda I$ is not-invertible with the determinant. Row reduction does not help us here.

5.1.24 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has only the eigenvalue 1, as $Av = v$ for all v. So also every vector is an eigenvector for this value.

5.1.25 If $Ax = \lambda x$, then $x = A^{-1}(\lambda x) = \lambda A^{-1}x$, so $x/\lambda = A^{-1}x$, and $1/\lambda$ is an eigenvalue of A^{-1}.

5.1.29 Let v be the column vector with 1 in each entry. Then Av is the column vector consisting of row sums of A, but then this is sv, since each row sum is s. Thus $Av = sv$.

5.1.32 a.) $x_{k+1} = c_1\lambda^{k+1}u + c_2\mu^{k+1}v$. b.) $Ax_k = c_1\lambda^k Au + c_2\mu^k Av = c_1\lambda^{k+1}u + c_2\mu^{k+1}v = x_{k+1}$.

Section 5.2

5.2.4

$$\begin{vmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = (8-\lambda)(3-\lambda) - 6 = \lambda^2 - 11\lambda + 18 = (\lambda-2)(\lambda-9),$$

so the eigenvalues are 2 and 9.

5.2.10

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda)(7-\lambda)+1(0)(-2)+1(0)(0)-(-2)(5-\lambda)(1)-0(0)(3-\lambda)-(7-\lambda)(0)(1) =$$

5.2.19

$$\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

so

$$\det(A - 0I) = \det A = (\lambda_1 - 0)\cdots(\lambda_n - 0) = \lambda_1\lambda_2\cdots\lambda_n.$$