Math 54 – HW 5 solutions

Section 4.1

4.1.1
(a) Let \( u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \) and \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \). Then note that as \( u_1 \geq 0 \) and \( v_1 \geq 0 \), we have that \( u_1 + v_1 \geq 0 \). By same logic \( u_2 + v_2 \geq 0 \).

Thus \( u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \in V \).

(b) Take \( u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and \( c = -1 \). Then \( cu = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) and thus \( c \notin V \). Note that the choice of \( u \) is not really important, and any negative value of \( c \) would have done.

4.1.2
a) Again let \( u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \). Then note that as \( u \in W \), we have \( u_1 * u_2 \geq 0 \). Also as \( c \) is a real number, \( c^2 \geq 0 \). Thus we have
\[
(c^2)u_1 * u_2 \geq 0 \Rightarrow (cu_1)(cu_2) \geq 0 \Rightarrow cu = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \in W
\]

(1)

b) Let \( u = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) and \( v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Note that as \( 0 * (-1) = 0 \) and \( 0 * 1 = 0 \), \( u \) and \( v \) are both in \( W \).

But \( u + v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) and as \( 1 * (-1) = -1 \notin 0 \), we have \( u + v \notin W \).

4.1.5 Indeed. Let \( Q = \{ \text{All polynomials of the form } at^2 \text{ for some } a \in \mathbb{R} \} \). And let \( p \) and \( q \) be two such polynomials and let \( c \) be some arbitrary real number. Then if \( p(t) = pt^2 \) and \( q(t) = qt^2 \) we have \( (p + cq)(t) = p(t) + cq(t) = pt^2 + cq(t) = (p_1 + cq_1)t^2 \). Thus as \( p + cq \in Q \) and \( Q \subset P_n \), we have that \( Q \) is a vector subspace of \( P_n \).

Note: \( p - p = 0 \). Thus our proof has implicitly proved that the zero vector is in \( Q \).

4.1.6 No. Simply note that the polynomial which is zero everywhere is not in the required set. As every vector space must contain a 0 element and the 0 element for \( P_n \) is the polynomial which is zero everywhere, the required set cannot be a vector subspace of \( P_n \).

4.1.8 Indeed. As before let \( Q = \{ \text{All polynomials } p \text{ such that } p(0) = 0 \} \). And let \( p \) and \( q \) be two such polynomials and let \( c \) be some arbitrary real number. Then \( (p + cq)(0) = p(0) + cq(0) = 0 + c * 0 = 0 \). Thus as \( p + cq \in Q \) and \( Q \subset P_n \), we have that \( Q \) is a vector subspace of \( P_n \).
4.1.11 Note that 
\[
\begin{bmatrix}
2b + 3c \\
- b \\
2c
\end{bmatrix} = b\begin{bmatrix}
2 \\
-1 \\
0
\end{bmatrix} + c\begin{bmatrix}
3 \\
0 \\
2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
2b + 3c \\
- b \\
2c
\end{bmatrix} \in \text{Span}\{\begin{bmatrix}2 \\ -1 \\ 0\end{bmatrix}, \begin{bmatrix}3 \\ 0 \\ 2\end{bmatrix}\}.
\]

On the other hand any element of \(\text{Span}\{\begin{bmatrix}2 \\ -1 \\ 0\end{bmatrix}, \begin{bmatrix}3 \\ 0 \\ 2\end{bmatrix}\}\) looks like \(x_1\begin{bmatrix}2 \\ -1 \\ 0\end{bmatrix} + x_2\begin{bmatrix}3 \\ 0 \\ 2\end{bmatrix}\) and is thus an element of \(W\). Thus \(W = \text{Span}\{\begin{bmatrix}2 \\ -1 \\ 0\end{bmatrix}, \begin{bmatrix}3 \\ 0 \\ 2\end{bmatrix}\}\).

As \(\begin{bmatrix}2 \\ -1 \\ 0\end{bmatrix}\) and \(\begin{bmatrix}3 \\ 0 \\ 2\end{bmatrix}\) are both elements of \(\mathbb{R}^3\), by Theorem 1 of the section we have that \(W\) is a vector subspace of \(\mathbb{R}^3\).

4.1.20

(a) To prove the claim we would need the following facts:

1) The sum of two continuous function is continuous.

2) A continuous function when multiplied by a scalar remains continuous.

3) The function which is identically zero everywhere is a continuous function.

Note: fact (3) is simply a consequence of facts (1) and (2) and is thus redundant.

(b) Let \(Q = \{f \in C[a,b]: f(a) = f(b)\}\). Let \(f\) and \(g\) be two arbitrary elements of \(Q\) and let \(c\) be any real number. Then \((f + cg)(a) = f(a) + cg(a) = f(b) + cg(b) = (f + cg)(b)\). Thus \(f + cg \in Q\) and as \(Q \subset C[a,b]\), we have that \(Q\) is a vector subspace.

4.1.21 The easiest way to prove this is to show that \(H\) is a span of elements of \(M_{2\times2}\), for then by Theorem 1, we would have that \(H\) is a subspace of \(M_{2\times2}\).

Indeed \(\begin{bmatrix}a & b \\ 0 & d\end{bmatrix} = a\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0 \\ 0 & 1\end{bmatrix}\). Thus \(H = \text{Span}\{\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 \\ 0 & 1\end{bmatrix}\}\) and hence \(H\) is a vector subspace of \(M_{2\times2}\).

4.1.22 Let \(F\) and \(G\) be two matrices in \(H\) and let \(c\) be some real number. As always to prove that \(H\) is a vector subspace of \(M_{2\times4}\) is is enough to show that \(F + cG \in H\). Now note that \((F + cG)A = FA + cGA = 0 + c*0\) (\(FA = GA = 0\) as \(F\) and \(G\) are in \(H\)). Hence \(F + cG \in H\). Hence proved.

4.1.23
a False. Consider the function \( f(t) = t^2 \). Then \( f(0) = 0 \) but \( (f + f)(t) = 2t^2 \neq t^2 \). Thus \( f \) is not the zero vector.

b False. Consider any vector of \( \mathbb{R}^4 \) namely \((1, 1, 1, 1)\). Its not a element of a 3-dimensional space.

c False. Consider the case when \( V = \mathbb{R}^2 \) and \( H \) just contains two elements, in particular \( H = \{(0, 0), (1, 0)\} \). (A vector subspace would require \((-1, 0)\) to also be in \( H \)).

d True. This is explicitly written and proved in the first line after the book defines vector subspaces (pg. 181).

e True.

4.1.31 Consider any element \( w \in Span\{u, v\} \). Then there exists scalars \( a \) and \( b \) such that \( w = au + bv \).

Now from Property (3) in the definition of subspace, \( u \in H \Rightarrow au \in H \). Again by same logic \( v \in H \Rightarrow bv \in H \).

Then by Property (2), as \( au \) and \( bv \) are in \( H \), we have that \( w = au + bv \in H \).

\( \therefore \) \( Span\{u, v\} \subseteq H \) (since we just showed that any element of the former is also in the latter).

4.1.32 As always let \( u \) and \( v \) be two elements of \( H \cap K \), and let \( c \) be some arbitrary scalar. Then by property of intersection, \( u \) and \( v \) are also elements of \( H \). Therefore as \( H \) is a subspace, \( u + cv \in H \). By same argument as \( K \) is also a subspace, \( u + cv \in K \).

Therefore, \( u + cv \in H \cap K \). This proves that \( H \cap K \) is a vector subspace of \( V \).

Let \( H = Span\{(1, 0)\} \) a.k.a. the "X-axis" and let \( K = Span\{(0, 1)\} \) a.k.a. the "Y-axis". Then clearly \((1, 1) = (1, 0) + (0, 1) \notin H \cup K \). Thus \( H \cup K \) is not a vector subspace.