Math 54 – HW 5 solutions

Section 4.1

4.1.1

(a) Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then note that as $u_1 \geq 0$ and $v_1 \geq 0$, we have that $u_1 + v_1 \geq 0$. By same logice $u_2 + v_2 \geq 0$.

Thus $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \in V$

(b) Take $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $c = -1$. Then $cu = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and thus $c \notin V$. Note that the choice of u is not really important, and any negative value of c would have done.

4.1.2

a Again let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Then note that as $u \in W$, we have $u_1 * u_2 \geq 0$. Also as c is a real number, $c^2 \geq 0$. Thus we have

$$(c^2)u_1 * u_2 \geq 0 \Rightarrow (cu_1)(cu_2) \geq 0 \Rightarrow cu = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \in W \qquad (1)$$

b Let $u = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that as $0 * (-1) = 0$ and $0 * 1 = 0$, u and v are both in W.

But $u + v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and as $1 * (-1) = -1 \ngeq 0$, we have $u + v \notin W$

4.1.5 Indeed. Let $Q = \{$All polynomials of the form at^2 for some $a \in \mathbb{R}\}$. And let p and q be two such polynomials and let c be some arbitrary real number. Then if $p(t) = p_1 t^2$ and $q(t) = q_1 t^2$ we have $(p + cq)(t) = p(t) + cq(t) = p_1 t^2 + cq_1 t^2 = (p_1 + cq_1)t^2$. Thus as $p + cq \in Q$ and $Q \subset P_n$, we have that Q is a vector subspace of P_n.

Note : $p - p = 0$. Thus our proof has implicitly proved that the zero vector is in Q.

4.1.6 No. Simply note that the polynomial which is zero everywhere is not in the required set. As every vector space must contain a 0 element and the 0 element for P_n is the polynomial which is zero everywhere, the required set cannot be a vector subspace of P_n

4.1.8 Indeed. As before let $Q = \{$All polynomials p such that $p(0) = 0\}$. And let p and q be two such polynomials and let c be some arbitrary real number. Then $(p + cq)(0) = p(0) + cq(0) = 0 + c * 0 = 0$. Thus as $p + cq \in Q$ and $Q \subset P_n$, we have that Q is a vector subspace of P_n.

4.1.11 Note that $\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} = \begin{bmatrix} 2b \\ -b \\ 0 \end{bmatrix} + \begin{bmatrix} 3c \\ 0 \\ 2c \end{bmatrix} = b \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

$\therefore \begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} \in Span\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \}$

On the other hand any element of $Span\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \}$ looks like $x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2x_1+3x_2 \\ -x_1 \\ 2x_2 \end{bmatrix}$ and is thus an element of W. Thus $W = Span\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \}$.

As $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ are both elements of $\mathbb{R}^3$, by Theorem 1 of the section we have that W is a vector subspace of $\mathbb{R}^3$.

4.1.20

(a) To prove the claim we would need the following facts :

1) The sum of two continuous function is continuous.

2) A continuous function when multiplied by a scalar remains continuous.

3) the function which is identically zero everywhere is a continuous function.

Note : fact (3) is simply a consequence of facts (1) and (2) and is thus redundant.

(b) Let $Q = \{f \in C[a,b] : f(a) = f(b)\}$. Let f and g be two arbitrary elements of Q and let c be any real number. Then $(f+cg)(a) = f(a)+cg(a) = f(b)+cg(b) = (f+cg)(b)$. Thus $f+cg \in Q$ and as $Q \subset C[a,b]$, we have that Q is a vector subspace.

4.1.21 The easiest way to prove this is to show that H is a span of elements of $M_{2\times2}$, for then by Theorem 1, we would have that H is a subspace of $M_{2\times2}$.

Indeed $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $H = Span\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$ and hence H is a vector subspace of $M_{2\times2}$.

4.1.22 Let F and G be two matrices in H and let c be some real number. As always to prove that H is a vector subspace of $M_{2\times4}$ is is enough to show that $F+cG \in H$. Now note that $(F+cG)A = FA+cGA = 0+c*0$ ($FA = GA = 0$ as F and G are in H). Hence $F+cG \in H$. Hence proved.

4.1.23

a False. Consider the function $f(t) = t^2$. Then $f(0) = 0$ but $(f + f)(t) = 2t^2 \neq t^2$. Thus f is not the zero vector.

b False. Consider any vector of $\mathbb{R}^4$ namely $(1, 1, 1, 1)$. Its not a element of a 3-dimensional space.

c False. Consider the case when $V = \mathbb{R}^2$ and H just contains two elements, in particular $H = \{(0, 0), (1, 0)\}$. (A vector subspace would require $(-1, 0)$ to also be in H).

d True. This is explicitly written and proved in the first line after the book defines vector subspaces (pg. 181).

e True.

4.1.31 Consider any element $w \in Span\{u, v\}$. Then there exists scalars a and b such that $w = au + bv$.

Now from Property (3) in the definition of subspace, $u \in H \Rightarrow au \in H$. Again by same logic $v \in H \Rightarrow bv \in H$.

Then by Property (2), as au and bv are in H, we have that $w = au + bv \in H$.

$\therefore Span\{u, v\} \subseteq H$ (since we just showed that any element of the former is also in the latter).

4.1.32 As always let u and v be two elements of $H \bigcap K$, and let c be some arbitrary scalar. Then by property of intersection, u and v are also elements of H. Therefore as H is a subspace, $u + cv \in H$. By same argument as K is also a subspace, $u + cv \in K$. Therefore, $u + cv \in H \bigcap K$. This proves that $H \bigcap K$ is a vector subspace of V.

Let $H = Span\{(1, 0)\}$ a.k.a. the "X-axis" and let $K = Span\{(0, 1)\}$ a.k.a. the "Y-axis". Then clearly $(1, 1) = (1, 0) + (0, 1) \notin H \cup K$. Thus $H \cup K$ is not a vector subspace.