Math 54 – HW 4 solutions

Section 2.2

2.2.10

(a) False: Recall that performing a series of elementary row operations A is equivalent to multiplying A by a series of elementary matrices. Suppose that $E_1, \dots, E_k$ are the elemenatry matrices corresponding to the row operations used to reduce A to I. So as we saw in class, $A^{-1} = E_k \cdots E_1$. Thus applying the same elementary row operations to A^{-1} gives us $E_k \cdots E_1 A^{-1} = A^{-1}A^{-1} = A^{-2}$, which in general is not equal to I.

(b) True: Since $AA^{-1} = A^{-1}A = I$, A is the inverse of A^{-1}.

(c) False: The inverse of a product of invertible matrices is the product of their inverses in the reverse order. More concretely, if $A_1, A_2, \dots, A_m$ are invertible $n \times n$ matrices then $(A_1 A_2 \cdots A_m)^{-1} = A_m^{-1} \cdots A_2^{-1} A_1^{-1}$.

(d) True: For each $j \le n$ let $\mathbf{v}_j$ be a solution to $A\mathbf{x} = \mathbf{e}_j$. Define B to be the matrix whose j^{th} column is $\mathbf{v}_j$. Then $AB = [A\mathbf{v}_1 \dots A\mathbf{v}_n] = I$. Since A is square, this implies that A is invertible (and B is its inverse). Alternatively, we can observe that since every vector in $\mathbb{R}^n$ can be written as a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$ the given assumption implies that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$. Thus the linear transformation defined by A is onto and since A is square it is therefore invertible.

(e) True: This is the content of Theorem 7 in the textbook. This follows from the observations that performing an elemenatry row operation is equivalent to multiplying by an elementary matrix and that elementary matrices are invertible.

2.2.12 Using the fact that A is invertible, we may multiply both sides of $AD = I$ by A^{-1} and then use basic facts of matrix algebra to simplify:

$$\begin{aligned} AD &= I \\ A^{-1}(AD) &= A^{-1}I \\ (A^{-1}A)D &= A^{-1} \\ D &= A^{-1} \end{aligned}$$

2.2.13 Since A is invertible, we may multiply both sides of $AB = AC$ by A^{-1} to get:

$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C \end{aligned}$$

In general, if A is not invertible this is not true. For instance consider

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

2.2.16 Note that $A = (AB)B^{-1}$ and thus A is the product of two invertible matrices. Since any product of two invertible matrices is itself invertible, A is invertible.

2.2.20

(a) Note that by multiplying both sides of $(A - AX)^{-1} = X^{-1}B$ by X we have $B = X(A - AX)^{-1}$. Since B is thus the product of two invertible matrices, B itself is invertible (see problem 16).

(b) Let's try to solve this as we would any algebraic equation: group together and isolate all occurences of X.

$$\begin{aligned} (A - AX)^{-1} &= X^{-1}B \\ X(A - AX)^{-1} &= B \\ X &= B(A - AX) \\ X &= BA - BAX \\ X + BAX &= BA \\ (I + BA)X &= BA \end{aligned}$$

We would next like multiply both sides by the inverse of $(I + BA)$ and reach an expression for X. To verify that $(I + AB)$ is invertible, note that by the last equation above, $(I + BA) = BAX^{-1}$. By assumption A and X^{-1} are invertible and by part (a), B is invertible. So $(I + BA)$ is the product of three invertible matrices and thus invertible. So we are indeed justified in multiplying by the inverse of $(I + BA)$ and thus we have $X = (I + BA)^{-1}BA$.

If you dislike the appearance of I in this expression, note that $(I + BA) = B(B^{-1} + A)$ and thus an alternative expression for X is $X = (B(B^{-1} + A))^{-1}BA = (B^{-1} + A)^{-1}B^{-1}BA = (B^{-1} + A)^{-1}A$.

2.2.22 If A is invertible then it is row equivalent to the $n \times n$ identity matrix I_n. Since I_n has a pivot in every row, so does A. Thus the columns of A span $\mathbb{R}^n$.

2.2.24 If the system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$ then A must have a pivot in every row. Since A is $n \times n$ this implies that A also has a pivot in every column (since there is at most one pivot per column, only n columns total and at least n pivots since each row has one). But the only $n \times n$ matrix in RREF with a pivot in every row and column is I_n. So A is row equivalent to I_n and thus invertible.

2.2.30

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & -1 & -4/3 & 1 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & 1 & 4/3 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & -7/3 & 2 \\ 0 & 1 & 4/3 & -1 \end{array}\right]$$

At this point we see that the original matrix is in fact invertible because we have row reduced it to the identity matrix and thus its inverse is

$$\begin{bmatrix} -7/3 & 2 \\ 4/3 & -1 \end{bmatrix}$$

2.2.32

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -7 & -2 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{array}\right]$$

Since the above row reduction shows that A is not row equivalent to the identity matrix (since it only has 2 pivots), A is not invertible.

Section 2.3

2.3.2 We reduce to REF and then check if there are two pivot positions.

$$\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/2 \\ 6 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Since the second row does not have a pivot, the matrix is not invertible.

2.3.5 Notice that when we apply elementary row operations to the given matrix, the second column will always stay all zeros. Thus the second column does not contain a pivot position and so the matrix is not invertible.

2.3.8 The matrix is already almost in REF and it is clear that when it is put in REF there will be a pivot in every column. Thus the matrix is invertible.

2.3.12

(a) True: By the invertible matrix theorem (theorem 8), since A is square and $AD = I$, A is invertible. Thus since $AD = I$, $D = A^{-1}$ and so $DA = I$.

(b) False: This only holds if A is invertible. As a counterexample, take A to be the matrix where all entries are zero.

(c) True: Since A is square we can apply the invertible matrix theorem. So if the columns are linearly independent then the matrix is invertible and thus its columns span $\mathbb{R}^n$.

(d) False: If A is invertible then $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one but $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$.

(e) False: This is only true if the transformation $x \mapsto Ax$ is one-to-one (and since A is square that means it is only true if A is invertible). As a counterexample, take A to be the matrix where all entries are zero and take $\mathbf{b}$ to be the zero vector. Then there is a solution to $A\mathbf{x} = \mathbf{b}$ but it is certainly not unique.

2.3.15 It is not possible since it contradicts the invertible matrix theorem (theorem 8). It is also possible to see this directly: if the columns of A don't span $\mathbb{R}^4$ then for some $\mathbf{b} \in \mathbb{R}^4$ the equation $A\mathbf{x} = \mathbf{b}$ has no solution so A cannot be row equivalent to the identity matrix.

2.3.21 The columns cannot span $\mathbb{R}^n$. The fact that $C\mathbf{u} = \mathbf{v}$ has more than one solution for some $\mathbf{v} \in \mathbb{R}^n$ means that the transformation $\mathbf{x} \mapsto C\mathbf{x}$ is not one-to-one. Hence the invertible matrix theorem implies that C is not invertible and therefore by the invertible matrix theorem the columns of C don't span $\mathbb{R}^n$.

2.3.28 By the invertible matrix theorem, to check that B is invertible, it suffices to check that the columns of B are linearly independent, which is equivalent to checking that there is no nontrivial solution to the homogeneous equation $B\mathbf{x} = \mathbf{0}$. Let $\mathbf{v} \in \mathbb{R}^n$ be a vector such that $B\mathbf{v} = \mathbf{0}$. So $(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0} = \mathbf{0}$. Thus $\mathbf{v}$ is a solution to the homogeneous equation $(AB)\mathbf{x} = \mathbf{0}$. Since AB is invertible, by the invertible matrix theorem we must have $\mathbf{v} = \mathbf{0}$. Thus we have shown that the only solution to $B\mathbf{x} = \mathbf{0}$ is the trivial one.

2.3.36 If a linear transformation T has this property then by definition it is not one-to-one. Thus the standard matrix, A, for T is not invertible. So by theorem 9 T is not invertible. Therefore by the invertible matrix theorem the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not onto. Since this linear transformation is just T, this means T is not onto.

Section 2.6

2.6.2 The vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ are both in the set pictured, but their sum, $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, is not.

2.6.5 We need to determine if $\mathbf{w}$ is in the span of $\mathbf{v_1}$ and $\mathbf{v_2}$. As we saw in chapter 1, we can do this by solving a system of linear equations. We proceed by row reduction:

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 3 & -3 & -3 \\ -4 & 7 & 10 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Since there is no pivot in the rightmost column of the above augmented matrix, the system is consistent and thus $\mathbf{w}$ is in the subspace generated by $\mathbf{v_1}$ and $\mathbf{v_2}$.

2.6.7

(a) Three vectors.

(b) Col A is all linear combinations of three nonzero vectors, so it contains infinitely many vectors.

(c) To check if $\mathbf{p}$ is in Col A we need to check if the system $A\mathbf{x} = \mathbf{p}$ has a solution. As usual, we row reduce:

$$\left[\begin{array}{ccc|c} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3/2 & -2 & 3 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3/2 & -2 & 3 \\ 0 & -4 & -10 & 14 \\ 6 & -7 & -7 & 11 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3/2 & -2 & 3 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3/2 & -2 & 3 \\ 0 & 1 & 5/2 & -7/2 \\ 0 & 2 & 5 & -7 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3/2 & -2 & 3 \\ 0 & 1 & 5/2 & -7/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the final matrix above is in REF, we can see that the rightmost column of the augmented matrix does not have a pivot. Therefore $\mathbf{p}$ is in Col A.

2.6.11 Since the matrix is 3×4, Nul A is a subspace of $\mathbb{R}^4$ and Col A is a subspace of $\mathbb{R}^3$.

2.6.13 For a nonzero vector in Col A we can simply use a nonzero column of A, such as $\left[\begin{smallmatrix} 3 \\ -9 \\ 9 \end{smallmatrix}\right]$ (though there are *many* other valid answers).

For a nonzero vector in Nul A we need to find a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. So as usual we row reduce.

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2/3 & 1/3 & 5/3 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2/3 & 1/3 & 5/3 \\ 0 & 2 & 4 & 22 \\ 9 & 2 & -5 & 1 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 2/3 & 1/3 & 5/3 \\ 0 & 2 & 4 & 22 \\ 0 & -4 & -8 & -14 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2/3 & 1/3 & 5/3 \\ 0 & 1 & 2 & 11 \\ 0 & -4 & -8 & -14 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & -17/3 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 30 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now that the matrix is in RREF we can easily read off solutions to the homogeneous equation. They consist of vectors of the form

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_2$$

Choosing $x_2 = 1$, we find that one vector in Nul A is

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

2.6.18 To check if the vectors are a basis for $\mathbb{R}^3$ we need to check that they span $\mathbb{R}^3$ and are linearly independent. So as usual we row reduce

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & -1 & 1 \\ -3 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & -4 & -4 \\ 0 & 11 & 11 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 11 & 11 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix above is in REF and so it is easy to see that it does not have a pivot in every column and thus the vectors are not linearly independent (they also don't span $\mathbb{R}^n$ since there is not a pivot in every row). So they do not form a basis for $\mathbb{R}^n$.

2.6.22

(a) False: This is a necessary condition for H to be a subspace, but far from sufficient. For instance the set $\{\mathbf{0}, \mathbf{e}_1\}$ is a subset of $\mathbb{R}^n$ that contains the zero vector but is not a subspace.

(b) False: In general, performing row operations on a matrix changes the column space. For instance, consider the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. When we put this matrix in echelon form we get $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. And the column space of these two matrices is different (the column space of the first is all vectors whose first coordinate is zero while the column space of the second is all vectors whose second coordinate is zero).

(c) True: Adding two linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ yields a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$, as does multiplying by a scalar.

(d) False: This is only true if $\mathbf{y} \in H$. Note that since H is a subspace, $\mathbf{0} \in H$ but if $\mathbf{y} \notin H$ then $\mathbf{0} + \mathbf{y} = \mathbf{y}$ is not in H.

(e) False: The column space is the span of the columns of A. To see why this is different, note that $\mathbf{0}$ is always in the column space of A, but $\mathbf{0}$ is only a solution to $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \mathbf{0}$ (but note that even if $\mathbf{b} = \mathbf{0}$, it is still not in general the case that the column space of A is the set of solutions to $A\mathbf{x} = \mathbf{b}$).

2.5.24 To find a basis for Col A we simply take the columns of A that correspond to pivot columns in the echelon form. I.e. one basis for Col A is:

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$$

To find a basis for Nul A, we need to find the solutions to $A\mathbf{x} = \mathbf{0}$. We already have the matrix in REF, but let's put it into RREF to make things easier.

$$\begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions to the homogeneous equation consist of vectors of the form

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_4$$

So one basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

2.6.27

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 100 \end{bmatrix}$$

2.6.28

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 100 \end{bmatrix}$$

2.6.34 If the columns of A form a basis for $\mathbb{R}^m$ then in particular they span $\mathbb{R}^m$, so there is a pivot position in every row of A, and they are linearly independent so there is a pivot position in every column of A. Thus A must be a square matrix– i.e. $n = m$ (and in fact A is also invertible).

Section 2.7

2.7.3 We need to find c_1, c_2 such that $\mathbf{x} = c_1\mathbf{b_1} + c_2\mathbf{b_2}$. So we row reduce

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -3 & 5 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 7/2 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

So the answer is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2.7.5 As in problem 3, we row reduce

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 4 & -7 & 9 \\ -3 & 5 & -7 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

So the answer is

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

2.7.7 Based on the grid drawn in the problem, it looks like we need to add $\mathbf{b}_1$ twice and then go in the opposite direction of $\mathbf{b_2}$ once (i.e. subtract $\mathbf{b}_2$). So our guess for $[\mathbf{w}]_{\mathcal{B}}$ is

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Similarly, it looks like we can reach $\mathbf{x}$ by adding one and a half copies of $\mathbf{b}_1$ and half of $\mathbf{b}_2$. Thus our guess for $[\mathbf{w}]_{\mathcal{B}}$ is

$$\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}.$$

We can confirm this by multiplying the above vector by the matrix whose columns are $\mathbf{b}_1$ and $\mathbf{b}_2$ and checking that the result is $\mathbf{x}$. And indeed,

$$\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3 \times 3/2 - 1/2 \\ 0 \times 3/2 + 2 \times 1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{x}$$

2.7.9 To find a basis for Col A we simply take the columns of A that have pivot positions. Namely

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 6 \\ 15 \end{bmatrix} \begin{bmatrix} -6 \\ 5 \\ 9 \\ 14 \end{bmatrix} \right\}$$

To find a basis for Nul A it is easiest to first put A all the way into RREF. When we do this we get the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus the set of solutions to $A\mathbf{x} = \mathbf{0}$ consists of vectors of the form

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2$$

and thus one basis for Nul A is

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2.7.16 Col A is **not** equal to $\mathbb{R}^3$ since the column space is a subset of $\mathbb{R}^4$, and not of $\mathbb{R}^3$.

Since A has three pivot columns, its rank is three. By the rank theorem (theorem 13), the rank of A plus the dimension of the null space of A must add up to the number of columns of A, which is 7. So the dimension of the null space is 4.

2.7.21 The solution space of $A\mathbf{x} = \mathbf{0}$ is exactly the null space of A. By the rank theorem (theorem 13), the rank plus the dimension of the null space of the matrix must be equal to the number of columns, which is 8. Since the rank is 7, the null space must have dimension 1.

2.7.23

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.7.24

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$