Math 54
Homework 3 Solutions
9/14

1 §1.8

1.8.2

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a/3 \\ b/3 \\ c/3 \end{bmatrix}.$$

■

1.8.8

The number of rows of a matrix is the size (dimension) of the space it maps to; the number of columns is the size of the space it maps from. Therefore A must be a 7×5 matrix. ■

1.8.12

We can solve this by row reducing the augmented matrix $[A|b]$, just like before, and seeing if there is a solution. If there is, b is in the range, and if not then it isn't.

$$\left[\begin{array}{cccc|c} 3 & 2 & 10 & -6 & -1 \\ 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ 1 & 4 & 10 & 8 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 3 \\ 3 & 2 & 10 & -6 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 1 & 4 & 10 & 8 & 4 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 3 \\ 0 & 2 & 4 & 6 & -10 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 4 & 8 & 12 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 2 & 4 & 6 & -10 \\ 0 & 4 & 8 & 12 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right].$$

Now we can see there's a pivot in the final column, hence there is no solution, meaning b is not in the range of the transformation. ■

1.8.14

T scales each vector by a factor of 2. ■

1.1.16

T projects each vector onto the y-axis and scales that by a factor of 2. ■

1.8.17

$$\begin{aligned} T(2u) &= 2T(u) = 2 \cdot \left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 8 \\ 2 \end{smallmatrix}\right] \\ T(3v) &= 3T(v) = 3 \cdot \left[\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -3 \\ 9 \end{smallmatrix}\right] \\ T(2u+3v) &= 2T(u) + 3T(v) = \left[\begin{smallmatrix} 8 \\ 2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} -3 \\ 9 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 \\ 11 \end{smallmatrix}\right]. \end{aligned}$$

■

1.8.22

(a) True. If the components of x are $[x_1, x_2, \ldots, x_n]$, and the columns of A are $v_1, v_2, \ldots, v_n$, then Ax is the linear combination $x_1v_1 + x_2v_2 + \ldots + x_nv_n$.
(b) True. This is because if A is a matrix, v and w are vectors of the right dimension, and r is a real number, then $A(rv) = r \cdot Av$ and $A(v + w) = Av + Aw$.
(c) False. This is an existence question. A uniqueness question would be "Is there only one vector v in $\mathbb{R}^n$ such that $Tv = c$?"
(d) True. This is the definition of a linear transformation.
(e) True. This follows from the definition mentioned in part (d): since $0 \cdot \vec{0} = \vec{0}$ ($\vec{0}$ is the zero vector, 0 is the real scalar), we see that $T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}$. ■

1.8.30

Since the vectors $v_1, \ldots, v_p$ span $\mathbb{R}^n$, we can write any element v of $\mathbb{R}^n$ in the form

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_p v_p.$$

Then

$$\begin{aligned} T(v) &= T(a_1 v_1 + a_2 v_2 + \ldots + a_p v_p) \\ &= T(a_1 v_1) + T(a_2 v_2) + \ldots + T(a_p v_p) \\ &= a_1 T(v_1) + a_2 T(v_2) + \ldots + a_p T(v_p) \\ &= a_1 \cdot \vec{0} + a_2 \cdot \vec{0} + \ldots + a_p \cdot \vec{0} = \vec{0}. \end{aligned}$$

In the second and third lines, we just used the definition of a linear operator. ■

1.8.31

If $\{v_1, v_2, v_3\}$ is linearly dependent, there must be some numbers a, b, c for which $av_1 + bv_2 + cv_3 = 0$ and not all a, b, c are 0. Hitting the equation with T, we get

$$\begin{aligned} T(av_1 + bv_2 + cv_3) &= T(0) = 0 \\ aT(v_1) + bT(v_2) + cT(v_3) &= 0. \end{aligned}$$

Since not all of a, b, c are 0, we have shown that $\{T(v_1), T(v_2), T(v_3)\}$ are linearly dependent (with the same coefficients as $\{v_1, v_2, v_3\}$). ■

1.8.32

There is an absolute value in the definition, so let's see what happens when we multiply by -1. We see $T(1,1) = (1-2, 1-4) = (-1,-3)$, and $T(-1,-1) = (-1-2, -1+4) = (-3, 3)$. But if T were linear, then $T(-1,-1) = -T(1,1)$, but it doesn't as we see. ■

2 §1.9

1.9.2

The standard matrix of T is the one whose columns are $T(e_1), T(e_2)$ and $T(e_3)$. Hence it is just

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}.$$

■

1.9.4

So $T(e_1) = e_1$ and $T(e_2) = 2e_1 + e_2$. Thus the standard matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

■

1.9.6

Look graphically at what is happening: if we rotate e_1 by $-3\pi/2$ radians, we get e_2, and if we rotate e_2, we get $-e_1$. That is to say, $T(e_1) = e_2, T(e_2) = -e_1$. Thus the matrix is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

■

1.9.9

Thinking graphically again: if we reflect e_1 nothing happens, and if we then rotate it by $-\pi/2$ we get $-e_2$. If we reflect e_2 over the x-axis we get $-e_2$, and if we rotate $-e_2$ we get $-e_1$. The matrix is then

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

■

1.9.19

First, let's calculate: $T(1,0,0) = (1,0); T(0,1,0) = (-5,1)$; and $T(0,0,1) = (4,-6)$. Hence the matrix that implements T is $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$. ■

1.9.22

If the first coordinate has to be 0, this means $2x_1 = x_2$. Plugging that in to the second coordinate, we get $-3x_1 + x_2 = -3x_1 + 2x_1 = -x_1$ and we want this to equal -1. Hence $x_1 = 1, x_2 = 2$ makes the first two coordinates work out. We don't have any more variables, so now the last thing we need to do is check that this solution also works for the third coordinate. Luckily it does: $2x_1 - 3x_2 = 2 - 6 = -4$ just as we wanted. So, the solution is $x = (1,2)$.

Alternately (and more systematically), the standard matrix for T is $\begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix}$, which gives the augmented matrix $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{array}\right]$. Row reducing gives $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -2 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 0 \end{array}\right]$. The final augmented matrix tells us the system is consistent, $x_2 = 2$, and then $x_1 = 1$. ■

1.9.26

(a,b) Recall from the section in the book that T is onto iff its standard matrix has a pivot in every row, and one-to-one if the standard matrix has a pivot in every column. Then from #2 the standard matrix is $\begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$. There are more columns than rows, so T cannot have a pivot in every column, hence cannot be one-to-one. However, subtracting 4 times the first row from the second row, row reduces the standard matrix, giving $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 17 & -20 \end{bmatrix}$, so there is a pivot in every row and hence T is onto. ■

1.9.33 (optional)

If B is some matrix, then Be_i, the product of B with the vector e_i, just gives the ith column of B. Since we are assuming $T(x) = Ax = Bx$ for all x, choosing $x = e_1, e_2, \ldots, e_n$ gives us that each column of A equals each column of B (i.e. the fact that $Ae_1 = Be_1$ tells us their first columns are the same, etc.) ■

1.9.35

If T is onto then $m \leq n$, and if T is one-to-one then $m \geq n$.

As stated in #20, T is onto iff its standard matrix has a pivot in every row. If T has more rows than columns, i.e. $m > n$, this cannot happen; so if T in fact is onto, then we need $m \leq n$.

Similarly, since T is one-to-one iff its standard matrix has a pivot in every column, and T has more columns than rows, then it cannot be one-to-one. So T being one-to-one means $m \geq n$. ■

1.9.36

Because it asks: for each w in the range of T, does there *exist* some vector v in the domain of T, such that $T(v) = w$.

Alternatively, let A be the standard matrix of T. We know from an earlier section that the equation $Av = w$ has a solution iff w is in the span of the columns of A. Hence, being onto asks: for each w in $\mathbb{R}^n$, does there *exist* a solution to $Av = w$? ■

3 §2.1

2.1.1

$$\begin{aligned}
-2A &= -2 \cdot \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}; \\
B - 2A &= \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} \text{ using the last line for } -2A \\
&= \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}; \\
CD &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 3 + 2 \cdot -1 & 1 \cdot 5 + 2 \cdot 4 \\ -2 \cdot 3 + 1 \cdot -1 & -2 \cdot 5 + 1 \cdot 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}.
\end{aligned}$$

The third expression, AC, is not defined, because the number of columns in A and the number of rows in C are unequal. ■

2.1.2

$$\begin{aligned}
A + 3B &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 23 & -15 & 2 \\ 7 & -17 & -7 \end{bmatrix} \\
DB &= \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 3 \cdot 7 + 5 \cdot 1 & 3 \cdot -5 + 5 \cdot -4 & 3 \cdot 1 + 5 \cdot -3 \\ -1 \cdot 7 + 4 \cdot 1 & -1 \cdot -5 + 4 \cdot -4 & -1 \cdot 1 + 4 \cdot -3 \end{bmatrix} \\
&= \begin{bmatrix} 26 & -35 & -12 \\ -3 & -11 & -13 \end{bmatrix}.
\end{aligned}$$

The quantity $2C - 3E$ is not defined, because C and E (and hence $2C$ and $-3E$) have different dimensions. The quantity EC is not defined because the number of columns of E is unequal to the number of rows of C. ■

2.1.10

Clearly $B \neq C$. Computing the rest,

$$\begin{aligned} AB &= \begin{bmatrix} 3 \cdot -1 + -6 \cdot 3 & 3 \cdot 1 + -6 \cdot 4 \\ -1 \cdot -1 + 2 \cdot 3 & -1 \cdot 1 + 2 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix} \\ AC &= \begin{bmatrix} 3 \cdot -3 + -6 \cdot 2 & 3 \cdot -5 + -6 \cdot 1 \\ -1 \cdot -3 + 2 \cdot 2 & -1 \cdot -5 + 2 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}, \end{aligned}$$

What this shows is that you can't divide (i.e. cancel) nonzero matrices like you do real numbers. ∎

2.1.12

A column of B should be a vector that A cancels: i.e. a solution to $Av = 0$. One such v is $(2, 1)$ (which you can find either by using row reduction to find a non-trivial solution, or just noticing that the first column times -2 equals the second column), and any multiple of this also works: say $(4, 2)$. Hence $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ works. ∎

2.1.15

(a) False: multiplication of column vectors is not defined.
(b) False: switching around A and B in the sentence makes it true.
(c) True: the distributive property continues to hold for matrix multiplication, by linearity.
(d) True: it doesn't matter whether we add the entries and then transpose them, or transpose the entries and then add them.
(e) False: transposes switch the order of products. ∎

2.1.18

It is all zeros. The third column of AB is the one where we go along the third *row* of A and the third column of B, multiply numbers and add as we go along. Since all the numbers from B here are 0, so will the entries in AB be. ∎

2.1.22

If we write B in terms of its columns, say $B = [v_1 v_2 \dots v_n]$, then $AB = [Av_1 Av_2 \dots Av_n]$. So say the columns $\{v_1, \dots, v_n\}$ are linearly dependent: say $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ and not all the a_i's are 0. Then by linearity of matrix multiplication, $a_1 Av_1 + a_2 Av_2 + \dots + a_n Av_n = 0$, i.e. $\{Av_1, \dots, Av_n\}$ are linearly dependent as well. We said in the first sentence that these are the columns of AB, and this is what we initially set out to prove. ∎

2.1.23

If $Ax = 0$, then $CAx = C(Ax) = C0 = 0$. But $CA = I_n$, which means $CAx = I_n x = x$. Putting the two sentences together, this means $x = 0$.

If A had more columns than rows, the columns would be linearly dependent: after row reducing, there couldn't be a pivot in every column. By the last problem, this would mean that the columns of CA were linearly dependent; but clearly the columns of I_n are not (i.e. they are linearly *independent*), so if $CA = I_n$ then A cannot have more columns than rows. ■

2.1.31

As stated two problems ago, if we write A in terms of its column vectors, say $A = [v_1\ v_2\ \ldots\ v_n]$, where each v_i is in $\mathbb{R}^m$, then $I_m A = [I_m v_1\ I_m v_2\ \ldots\ I_m v_n]$. But $I_m v = v$ for any $v \in \mathbb{R}^m$, hence $I_m A = [v_1\ v_2\ \ldots\ v_n] = A$. ■

2.1.32

The column definition of AI_n says that the ith column of AI_n is a linear combination of the columns of A, whose coefficients are given by the ith column of I_n. But by definition, all of these coefficients are 0 except for the ith one, which is 1: hence rereading this definition, it says that the ith column of AI_n is the ith column of A. This then verbatim means that $AI_n = A$, as desired. ■